An Interval-based Framework for Clock Rate Synchronization*

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Abstract

This paper addresses the problem of synchronizing the rate of clocks in a fault-tolerant distributed system. Contrived to bring the rates of all non-faulty clocks in accordance, rate synchronization algorithms are very similar to usual "state" synchronization ones. Major differences, however, arise from the fact that the quantities to be synchronized are not directly accessible and that they do not proceed linearly with time. Relying on an interval-based paradigm, we introduce a basic system model and suitable building blocks for a generic rate synchronization algorithm that employs a convergence function. Our rigorous analysis of the mutual rate deviation (aka. consonance) and deviation from the ideal rate of 1 Sec/sec (aka. drift) reveals that the clock rate stability (i.e. the maximum rate change per unit of time) takes over the role of maximum hardware drift rate in traditional clock synchronization approaches.

Keywords: distributed systems, clock synchronization, drift, consonance, oscillator stability, rate intervals, convergence functions, clock validation.

1 Introduction

A clock is regarded as an entity that displays clock *state* T at real time t in some meaningful Newtonian frame, which can be modeled by a piecewise continuous function $C : t \mapsto T$. Commonly, clock synchronization aims to keep clock states together in the sense that there is some *precision* π bounding the clock state deviation between two different non-faulty clocks at any simultaneous real time, and some *accuracy* α bounding the deviation between corresponding clock states and real times on a single non-faulty clock. Maintaining these measures for an ensemble of clocks refers to the internal/external clock state synchronization problem, which has been studied thoroughly, see [Schn87], [ST87], [Cri89], [SWL90], [PR94] or [FC97] for instance.

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However, clock synchronization can be viewed in terms of clock speeds as well by considering the instantaneous clock *rate* $v(t) = dC(t)/dt$. By similarity, we are dealing with the internal/external clock rate synchronization problem, which can be phrased as follows:

(C) **Consonance Requirement**: There is some *consonance* γ such that $|v_p(t) - v_q(t)| \leq \gamma$ for all nodes p, q that are non-faulty up to real time t.

(D) **Drift Requirement**: There is some *drift* δ such that $|v_p(t) - 1| \leq \delta$ for all nodes p that are non-faulty up to real time t.

The benefits of clock rate synchronization are manyfold. Several distributed algorithms, when based on time-outs or round-trips, are content with rate synchronized clocks, cf. [Lis93]. Still, many applications require tight and robust state synchronized clocks, for instance distributed fault-tolerant real-time systems, cf. [KO87]. Most algorithms for clock state synchronization can be significantly improved w.r.t. achieving a better precision/accuracy or saving communication bandwidth when clock rates are additionally synchronized beyond the usual manufacturer's drift specification. This is due to a reduction of the precision/accuracy deterioration of freely running clocks between consecutive state resynchronization instants. Last but not least, rate synchronization can be useful to implement fault detection mechanisms and to improve initial clock synchronization.

Not much research has been conducted on the problem of clock rate synchronization. The only important contribution is the thesis [Mar84], which includes a presentation of an interval-based algorithm for constant-rate clocks combined with a round-trip method for rate measurement. Extending those ideas, we develop a comprehensive generic framework for analyzing convergence function-based rate synchronization algorithms that deal with both consonance and drift on top of a realistic system model. For an in-depth treatment the reader is referred to the technical report [Scho96].

This paper begins with an introduction to our system model in Section 2 concerning clocks and processors along with their means of communication. Section 3 is devoted to the building blocks for rate synchronization relying on an interval-based paradigm. The theory of a generic algorithm for internal/external rate synchronization is developed in Section 4, and Section 5 provides an analysis. Concluding remarks and future research issues close the paper.

2 System Modeling

We assume a distributed system of $n \geq 2$ *nodes* connected by a *communication subsystem*, where each node p hosts a *processor* $\mathcal{P}_p$ and a *clock* $\mathcal{C}_p$ driven by an *oscillator* $\mathcal{O}_p$. Figure 1 illustrates the components involved in steering a local clock, where a *clock state algorithm* (CSA) is concerned with precision/accuracy and a *clock rate algorithm* (CRA) with consonance/drift.

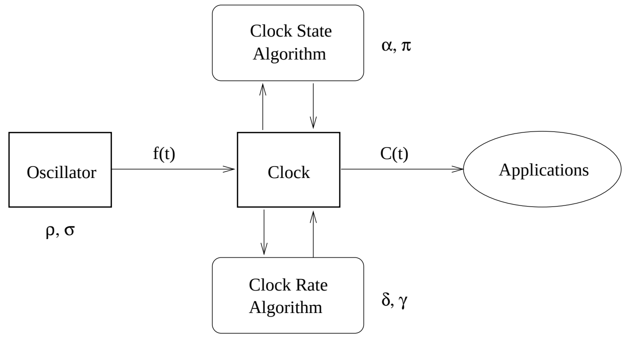

Figure 1: Clock Steering

The following system assumptions are only meaningful for non-faulty components. Since faults may occur in a real system, we specify prospective ones via an *(abstract) fault model* $\mathcal{F}$ in Section 4.2. Note that we do not focus on a specific one to keep our framework widely applicable.

2.1 Oscillator

The oscillator $\mathcal{O}_p$ indicates the progress of time with periodic ticks of *nominal frequency* f_p in [Hz]. During operation the *instantaneous frequency* $f_p(t)$ may change due to environmental influences and of the oscillator itself. A *maximum oscillator drift* $\rho_p^{\max}$ in [ppm], also referred as "hardware drift rate" in the clock synchronization literature, has to cover all deviations such that

$$\left|\frac{f_p(t)}{f_p} - 1\right| \leq \rho_p^{\max} \qquad \forall t \geq t_0. \tag{A1}$$

Typically, the manufacturer specifies an *initial oscillator drift* ρ_p in [ppm] for shipment along with further specifications, see [NIST90] or [V91] for more details.

For clock rate synchronization it is important that $f_p(t)$ does not alter too rapidly, otherwise rate resychronization will not be effective. Fortunately, experiments have shown that oscillators keep up their frequency to some extent, cf. [Tro94] or [Sche96]. Therefore, we are seeking for a characterization of $f_p(t)$ including deterministic influences (e.g. temperature, aging, pressure, humidity, magnetism, shock, power supply change, load impedance, radiation) and stochastic ones (e.g. white noise, flicker noise, random walk), cf. [A87]. For our purpose, we capture the frequency variation as follows: Suppose oscillator $\mathcal{O}_p$ has instantaneous frequency $f_p(t)$ at some real time t and $f_p(t+\Delta t)$ at $t+\Delta t$. We make the non-standard assumption that

$$\left|\frac{f_p(t+\Delta t)}{f_p(t)} - 1\right| \leq \sigma_p \Delta t \qquad \forall t \geq t_0, \tag{A2}$$

where parameter σ_p is called *oscillator stability* measured in [ppm/sec]. However, this characterization is only meaningful for particular durations $0 \leq \Delta t < \infty$. For very large ones, the term $\sigma_p \Delta t$ would become too pessimistic, since the specified maximum oscillator drift $\rho_p^{\max}$ prevents the frequency ratio on the long run from exceeding $1 \pm 2\rho_p^{\max}$. For very small durations, say a few oscillator periods $1/f_p$, glitches could violate the bounds, but we regard them as non-accumulating which can be confirmed by experiments (however, the variance of the frequency deviation becomes larger). A more advanced way of characterization would use an oscillator stability that is dependent on the duration, thus $\sigma_p(\Delta t)$.

By browsing through data sheets, we figured out that typical uncompensated quartz oscillators (XO) have $\rho \approx 10^{-5}$ and $\sigma \approx 10^{-7}$ sec^{-1}, temperature compensated ones (TCXO) have $\rho \approx 10^{-6}$ and $\sigma \approx 10^{-9}$ sec^{-1}, and rubidium gas cell oscillators have $\rho \approx 10^{-10}$ and $\sigma \approx 10^{-11}$ sec^{-1}.

2.2 Local Clock

The distinction between oscillator $\mathcal{O}_p$ and clock $\mathcal{C}_p$ as shown in Figure 1 is the hook to introduce rate synchronization for clocks. In the absence of a CRA, the oscillator is directly coupled to the clock, which is usually a simple hardware counting device. Thus, the clock rate becomes

$$v_p(t) = S_p f_p(t), \tag{A3}$$

where *coupling factor* S_p in [sec] is the constant $1/f_p$. In such an arrangement the clock drift δ_p equals the maximal oscillator drift $\rho_p^{\max}$.

It is vital to understand that rate synchronization demands to break up this rigid oscillator-clock coupling by getting a handle on S_p. In other words, even though $f_p(t)$ changes, the factor S_p should be set in such a way that the clock rate $v_p(t)$ remains approximately constant. Obviously, the realization of a local clock needs to provide some means to modify S_p, like our UTCSU-ASIC presented in [SSHL97]. The feasible range of coupling factor S_p is given by $1/f_p(1 \pm \rho_p^{\max})$, through which the clock drift δ_p of a non-faulty clock is now at most $2\rho_p^{\max}$. In the special case of $\sigma_p = 0$ we say the oscillator/clock is *stable*. Moreover, we define $\sigma_{\max} = \max\{\sigma_p \mid 1 \leq p \leq n\}$ as the uniform bound on the oscillator stability, and $\rho_{\max} = \max\{\rho_p^{\max} \mid 1 \leq p \leq n\}$ as the uniform bound on the maximal oscillator drift.

Additionally, we presume a precision $\pi_{\max}$ among non-faulty nodes as enforced by the coexisting CSA, thus

$$|C_p(t) - C_q(t)| \leq \pi_{\max} \qquad \forall t \geq t_0. \tag{A4}$$

The precision should be small enough to ensure that $\sigma_p \pi_{\max}$ can be neglected with respect to $\rho_p^{\max}$ for all $1 \leq p \leq n$. This facilitates a rounds-based CRA, which entails an inexpensive implementation and renders the analysis easier.

2.3 Processor

Each node is equipped with a processor $\mathcal{P}_p$ that executes the CRA. All computations are essentially periodic and in terms of the execution speed, memory and organization of the processor, we require that there is an upper bound $\eta_{\max}$ in [sec] on the total execution time (A5).

2.4 Communication Subsystem

Nodes communicate with each other via a packet-based communication subsystem that provides an (un)reliable broadcast primitive. It can either be a fully connected point-to-point network or a broadcast-type network, where the *maximum broadcast latency* $\lambda_{\max}$ in [sec] delimits the time between initiating and actually sending a packet (A6), and the *maximum broadcast operation delay* $\tau_{\max}$ in [sec] delimits the time to send out all packets (A7). When no faults occur, a transmission is assumed to exhibit synchronous behavior, i.e., a packet from node q to node $p \neq q$ experiences a delay $\Delta t'_{p,q}$ satisfying

$$\Delta t_{p,q} - \epsilon^-_{p,q} \leq \Delta t'_{p,q} \leq \Delta t_{p,q} + \epsilon^+_{p,q}, \qquad \text{(A8)}$$

where $\Delta t_{p,q}$ in [sec] represents the deterministic part and $\epsilon^{\pm}_{p,q}$ in [sec] the delivery uncertainties. For simplicity we will work with $\epsilon_{\max} = \max\{\epsilon^-_{p,q} + \epsilon^+_{p,q} \mid 1 \leq p, q \leq n\}$ and $\Delta t_{\max} = \max\{\Delta t_{p,q} \mid 1 \leq p, q \leq n\}$. Values for $\epsilon_{\max}$ can either be fixed a priori or, preferable, measured on-line with round-trip oriented protocols. For a long haul network it is typically in the ms-range, and for LANs with a shared broadcast channel in the μs-range with appropriate hardware support, cf. [HSS97].

3 Building Blocks

This section presents the building blocks for clock rate synchronization by introducing notations and definitions as well as proving useful technical lemmas using $\mathcal{O}$-expressions[1] whose purpose is to specify the order of magnitude of terms that can be neglected in practice.

Our framework uses *asymmetric intervals* to capture the necessary information, see [SS97] or earlier [Schm94], [Lam87] and [MO83]. We denote them with bold capital letters such as $\mathbf{I} = [x, r, y]$, where $r \in \mathbb{R}^+_0$ is called the *reference point*, $x \in \mathbb{R}^+_0$ the *left length* and $y \in \mathbb{R}^+_0$ the *right length.* Such an asymmetric interval translates into a regular one $[r - x, r + y]$, where $(r - x)$ is the *left edge* and $(r + y)$ the *right edge*. In case that $x = y$ we also write $[r \pm x]$. An ordered set of intervals is written by bold calligraphic capital letters such as $\boldsymbol{\mathcal{I}}$. Operations on intervals are defined in a straightforward manner, like the sum of $\mathbf{I}_1 + \mathbf{I}_2 = [x_1 + x_2, r_1 + r_2, y_1 + y_2]$, the scalar product $s\mathbf{I}_1 = [sx_1, sr_1, sy_1]$, or the interval product $\mathbf{I}_1 \cdot \mathbf{I}_2 = [r_1 x_2 + r_2 x_1 - x_1 x_2, r_1 r_2, r_1 y_2 + r_2 y_1 + y_1 y_2]$. Furthermore, we define $\mathrm{ref}(\mathbf{I}) = r$, $\mathrm{align}(\mathbf{I}) = \mathbf{I} - [0, r, 0]$, $\mathrm{norm}(\mathbf{I}) = \mathbf{I}/\mathrm{ref}(\mathbf{I})$, $\mathrm{swap}(\mathbf{I}) = [y, r, x]$ and $||\mathbf{I}|| = x + y$.

3.1 Local Rate Intervals

For rate synchronization some measure of the clock rate v_p is needed, since it is not directly accessible. We can postulate an asymmetric interval with reference point $v_p r_p$ and sufficiently long left resp. right lengths $v_p \theta^-_p$ resp. $v_p \theta^+_p$ so that the ideal rate of 1 is included. Note that this introduces a relative expression of clocks rates. Dropping v_p leads to the definition of a *rate interval* $\mathbf{R}_p = [\theta^-_p, r_p, \theta^+_p]$, which contains enough information to run a clock rate algorithm.

[1] Recall that $f(x) = \mathcal{O}(g(x))$ for $x \to x_0$ if there is a positive constant M such that $|f(x)| \leq M|g(x)|$ for $x \to x_0$. We make frequent use of the asymptotic approximations like $(1 \pm x)^{-1} = 1 \mp x + \mathcal{O}(x^2)$ valid for $x \to 0$.

Definition 1 *A rate interval $\mathbf{R}_p = [\theta^-_p, r_p, \theta^+_p]$ is* correct *during a non-empty (real time) interval $\mathbf{T}$ iff*

$$1 \in v_p(t)\mathbf{R}_p \qquad \forall t \in \mathbf{T}.$$

In the special case that $\mathbf{R}_p$ has 1 as reference point, we call it a *local rate interval.* It can be maintained locally and possesses many useful properties. The following Lemma shows that there exists a correspondence between local rate intervals and clock drifts.

Lemma 1 *If clock $\mathcal{C}_p$ has clock drift δ_p then $\mathbf{R}_p = [\delta_p/(1 + \delta_p), 1, \delta_p/(1 - \delta_p)]$ is guaranteed to be a correct local rate interval. On the other hand, if $\mathbf{R}_p = [\theta^-_p, 1, \theta^+_p]$ is a correct local rate interval for clock $\mathcal{C}_p$ then the actual clock drift is at most $\delta_p = \max\left\{\theta^+_p/(1 + \theta^+_p), \theta^-_p/(1 - \theta^-_p)\right\}$.*

Proof By exploiting Definition 1. □

For further considerations it is important to relate an observable duration ΔT on a local clock with their real time counterpart Δt and vice versa. Our next Lemma establishes these relationships.

Lemma 2 *Given a clock $\mathcal{C}_p$ paced by an oscillator with stability σ_p. Let t_1 resp. t_2 be real times and $T_1 = C_p(t_1)$ resp. $T_2 = C_p(t_2)$ the corresponding clock states, where $t_1 \leq t_2$ and no resynchronization occurred in between. If clock $\mathcal{C}_p$ has rate $v_p(t_1)$ at t_1 then we have*

$$\begin{aligned} T_2 - T_1 &\leq v_p(t_1)\left((t_2 - t_1) + \frac{\sigma_p}{2}(t_2 - t_1)^2\right) \\ T_2 - T_1 &\geq v_p(t_1)\left((t_2 - t_1) - \frac{\sigma_p}{2}(t_2 - t_1)^2\right) \end{aligned}$$

and the converse

$$\begin{aligned} t_2 - t_1 &\leq \frac{T_2 - T_1}{v_p(t_1)} + \frac{\sigma_p(T_2 - T_1)^2}{2v_p^2(t_1)} + \mathcal{O}\left(\sigma_p^2(T_2 - T_1)^3\right) \\ t_2 - t_1 &\geq \frac{T_2 - T_1}{v_p(t_1)} - \frac{\sigma_p(T_2 - T_1)^2}{2v_p^2(t_1)} + \mathcal{O}\left(\sigma_p^2(T_2 - T_1)^3\right). \end{aligned}$$

Proof Sketch Applying the integral from t_1 to t_2 over $v_p(t)$ delivers $T_2 - T_1$, since we introduced the clock rate as the derivative of the time dependent function $C_p(t)$. In the absence of resynchronizations we get by virtue of (A2) the first part of the Lemma. For the second part we treat the resulting relation as a quadratic equation and choose the appropriate roots in order to find bounds on $t_2 - t_1$. □

Each node p maintains a local rate interval $\mathbf{R}_p$, which should remain correct as time proceeds. However, an oscillator stability $\sigma_p > 0$ calls for a proper enlargement.

Lemma 3 *Given a clock $\mathcal{C}_p$ paced by an oscillator with stability σ_p. If $\mathbf{R}_p$ is a correct local rate interval at T_1, then*

$$\mathbf{R}_p + \left[0 \pm \frac{\sigma_p(T_2 - T_1)}{1 - 2||\mathbf{R}_p||}\right] + \left[0 \pm \mathcal{O}\left(\sigma_p^2(T_2 - T_1)^2\right)\right]$$

is correct at $T_2 \geq T_1$, when no resynchronizations occurred in between.

Proof Sketch In the absence of resynchronizations between $T_1 = C_p(t_1)$ and $T_2 = C_p(t_2)$, assumption (A2) ensures that $|v_p(t_2)/v_p(t_1) - 1| \leq \sigma_p(t_2 - t_1)$. Then we make $v_p(t_1)$ explicit, bring in the observable duration $T_2 - T_1$ via Lemma 2, and simplify the denominator by using $v_p(t_1) \geq 1 - ||\mathbf{R}_p||/(1 - ||\mathbf{R}_p||)$ from Lemma 1. The resulting inequalities upon $v_p(t_1)$ are combined with $v_p(t_1)(1 - \theta^-_p) \leq 1 \leq v_p(t_1)(1 + \theta^+_p)$ induced by the correctness of $\mathbf{R}_p = [\theta^-_p, 1, \theta^+_p]$ at T_1, which eventually yields the above formula. □

3.2 Relative Rate Measurement

Any clock rate algorithm has to work on the grounds of relative rates between clock pairs since a local clock is unable to determine its rate by itself. We capture the relative rate of a remote clock against a local one with a *quotient rate interval* $\mathbf{Q}_{p,q}$. The indexing follows the rule that the first one denotes the local node and the second the remote one. Intuitively, such an interval quantifies how fast/slow a remote clock is in respect to the own one.

Definition 2 *Given two clocks $\mathcal{C}_p$ resp. $\mathcal{C}_q$ with their rates $v_p(t')$ resp. $v_q(t'')$ during the non-empty (real time) intervals $\mathbf{T}_p$ resp. $\mathbf{T}_q$. An asymmetric interval $\mathbf{Q}_{p,q}$ that satisfies*

$$\frac{v_q(t')}{v_p(t'')} \in \mathbf{Q}_{p,q} \qquad \forall t' \in \mathbf{T}_q,\ \forall t'' \in \mathbf{T}_p$$

is called a quotient rate interval of remote clock $\mathcal{C}_q$ during $\mathbf{T}_q$ against local clock $\mathcal{C}_p$ during $\mathbf{T}_p$.

We propose a simple protocol as illustrated in Figure 2 to obtain $\mathbf{Q}_{p,q}$. It is based on repeated message pairs, such that remote node q broadcasts periodically a message that contains its current clock state $T_q = C_q(t_q)$. Local node p records its current clock state $T_p = C_p(t_p)$ upon message arrival. Therefore, node p can extract clock states T_q, T_p from the latter message M_q, and $T_q^{\prec}$, $T_p^{\prec}$ from the earlier message $M_q^{\prec}$. With these four clock states and taking into account the transmission characteristics, node p can compute a quotient rate interval given by Lemma 5. For preparation, we need the following technical lemma.

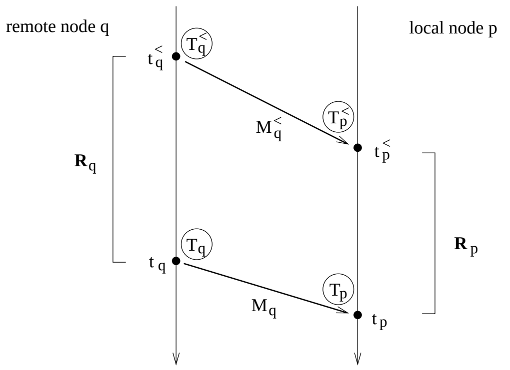

Figure 2: Protocol for Relative Rate Measuring

Lemma 4 *Given a clock $\mathcal{C}_p$ paced by an oscillator with stability σ_p and let t_1 resp. t_2 be real times and $T_1 = C_p(t_1)$ resp. $T_2 = C_p(t_2)$ the corresponding clock states, where $t_1 \leq t_2$. When no resynchronization occurred in between, then clock rate $v_p(t)$ satisfies*

$$v_p(t) \in \frac{T_2 - T_1}{t_2 - t_1}\left[1 \pm \frac{\sigma_p}{2}(t_2 - t_1)\right] + \left[0 \pm \mathcal{O}\left(\sigma_p^2 (t_2 - t_1)^2\right)\right]$$

for any $t \in [t_1, t_2]$.

Proof Sketch We have to consider two cases, either where the clock rate increases or decreases maximally during the real time duration $t_2 - t_1$ whereas the expired observable duration still amounts to $T_2 - T_1$. The actual clock rate lies somewhere between these extremes, which gives raise to bound $v_p(t)$ accordingly. □

Lemma 5 *Let $\mathcal{C}_p$ be the clock of local node p with stability σ_p and $\mathcal{C}_q$ be the clock of remote node q with stability σ_q. By executing the protocol in Figure 2 relying on assumption (A8), and provided that local rate interval $\mathbf{R}_p$ is correct during $[t_p^{\prec}, t_p]$ and $\mathbf{R}_q$ is correct during $[t_q^{\prec}, t_q]$, then interval*

$$\left[\frac{T_q - T_q^{\prec}}{T_p - T_p^{\prec}} \pm \left(\frac{(\sigma_p + \sigma_q)(T_q - T_q^{\prec})}{2(1 - ||\mathbf{R}_p||)} + \frac{\epsilon_{\max}(1 + ||\mathbf{R}_q||)}{T_p - T_p^{\prec}}\right)\right] +$$

$[0 \pm \mathcal{O}(\sigma_{\max}\epsilon_{\max} + \sigma_{\max}^2 (T_p - T_p^{\prec})^2 + \sigma_{\max}(T_p - T_p^{\prec})||\mathbf{R}_p||^2 + \epsilon_{\max}||\mathbf{R}_q||^2/(T_p - T_p^{\prec}))]$ *is a quotient rate interval during $[t_p^{\prec}, t_p]$ resp. $[t_q^{\prec}, t_q]$.*

Proof Sketch When $\Delta t_q = t_q - t_q^{\prec}$ is the involved real time duration at the sending node q and $\Delta t_p = t_p - t_p^{\prec}$ at the receiving node p, then we get $|\Delta t_q - \Delta t_p| \leq \epsilon_{\max}$ due to assumption (A8). From Lemma 4 we obtain expressions for minimum/maximum clock rates $v_p^{\min}$, $v_q^{\min}$, $v_p^{\max}$, $v_q^{\max}$ dependent on their real time measurement durations and put them in relation with the help of the above inequality. After several algebraic manipulations we are able to find a lower bound on $v_q^{\min}/v_p^{\max}$ and an upper bound on $v_q^{\max}/v_p^{\min}$ along with the $\mathcal{O}$-terms. □

Note that $||\mathbf{Q}_{p,q}||$ becomes a minimum for a certain measurement duration, i.e., there is an optimal choice of approximately $\sqrt{\epsilon_{\max}/\sigma_{\max}}$. Since rate measurements will usually take place less frequently than state resynchronizations, we have to care about interspersed state adjustments to get useful clock state differences. In fact, we need to maintain a running sum of state adjustments on both sides, denoted as U_p and U_q, irrespective of being applied instantaneously or by continuous amortization, cf. [SS97].

3.3 Remote Rate Intervals

A cornerstone of our framework is the transition of a correct local rate interval $\mathbf{R}_q$ at remote node q to a correct *remote rate interval* $\mathbf{R}_{p,q}$ at local node p, which in general does not have 1 as reference point. The switch of correctness between the two "rate worlds" involves the relative rate measurement through quotient rate interval $\mathbf{Q}_{p,q}$.

Lemma 6 *Let $\mathcal{C}_p$ and $\mathcal{C}_q$ be clocks driven by oscillators with stability σ_p and σ_q, respectively. As shown in Figure 2, the messages for the relative rate measurement are sent from node q at $T_q = C_q(t_q)$ resp. $T_q^{\prec} = C_q(t_q^{\prec})$ and received at node p at $T_p = C_p(t_p)$ resp. $T_p^{\prec} = C_p(t_p^{\prec})$. If $\mathbf{R}_q$ is a correct local rate interval at $t_q^{\prec}$ on remote node q, then*

$$\mathbf{R}_{p,q} = \mathbf{Q}_{p,q} \cdot \mathbf{R}_q$$

is a correct remote rate interval at t_p on local node p, where $\mathbf{Q}_{p,q}$ is a quotient rate interval during $[t_p^{\prec}, t_p]$ resp. $[t_q^{\prec}, t_q]$.

Proof The correctness of $\mathbf{R}_q = [\theta_q^-, 1, \theta_q^+]$ at $t_q^{\prec}$ reads as $v_q(t_q^{\prec})(1 - \theta_q^-) \leq 1 \leq v_q(t_q^{\prec})(1 + \theta_q^+)$. The quotient rate interval $\mathbf{Q}_{p,q}$ can be rewritten as $Q_{p,q}[u, 1, u]$ and by specialization of its meaning we get $Q_{p,q}(1 - u) \leq v_q(t_q^{\prec})/v_p(t_p) \leq Q_{p,q}(1 + u)$. Making $v_q(t_q^{\prec})$ explicit and plugging it into the above correctness relation, we obtain $v_p(t_p)Q_{p,q}(1 - u)(1 - \theta_q^-) \leq 1 \leq v_p(t_p)Q_{p,q}(1 + u)(1 + \theta_q^+)$, which is equivalent to $1 \in v_p(t_p)Q_{p,q}[u + \theta_q^- - u\theta_q^-, 1, u + \theta_q^+ - u\theta_q^+]$ and further $1 \in v_p(t_p)Q_{p,q}[u, 1, u] \cdot [\theta_q^-, 1, \theta_q^+] = v_p(t_p)(\mathbf{Q}_{p,q} \cdot \mathbf{R}_q)$. □

3.4 Consonance Intervals

For internal rate synchronization, consonance γ has to be expressed in terms of intervals. Returning to the definition of rate intervals, we introduce consonance by imposing distances upon their reference points.

Definition 3 *Given consonance interval $\boldsymbol{\gamma} = [\gamma^-, 0, \gamma^+]$ and an ensemble of clocks $\mathcal{C}_1, \ldots, \mathcal{C}_n$ with their respective rates $v_1(t), \ldots, v_n(t)$. An associated set of correct rate intervals $\boldsymbol{\mathcal{R}} = \{\mathbf{R}_1, \ldots, \mathbf{R}_n\}$ is $\boldsymbol{\gamma}$-consonant during a non-empty (real time) interval $\mathbf{T}$ iff*

$$\bigcap_{p=1}^{n} v_p(t)(\mathrm{ref}(\mathbf{R}_p) + \boldsymbol{\gamma}) \neq \emptyset \qquad \forall t \in \mathbf{T}.$$

As before, local rate intervals are playing a special role for establishing a relationship between the $\boldsymbol{\gamma}$-consonance property and the consonance γ of the ensemble.

Lemma 7 *Given an ensemble of clocks $\mathcal{C}_1, \ldots, \mathcal{C}_n$ each with drift δ. If such an ensemble has consonance γ, then a set $\boldsymbol{\mathcal{R}} = \{\mathbf{R}_1, \ldots, \mathbf{R}_n\}$ of correct local rate intervals is $\boldsymbol{\gamma}$-consonant for any $\boldsymbol{\gamma}$ satisfying $\boldsymbol{\gamma} \supseteq [0 \pm \gamma/(2(1-\delta))]$. On the other hand, if the set of correct local rate intervals $\boldsymbol{\mathcal{R}}$ is $\boldsymbol{\gamma}$-consonant, then the actual consonance is at most $\gamma = (1+\delta)||\boldsymbol{\gamma}||$.*

Proof By exploiting Definition 3. □

In order to have a better understanding of internal clock rate synchronization, we introduce the notion of an *internal global rate* $\omega(t)$ as analogue to an internal global time as advocated in [SS97]. The idea is to define $\omega(t)$ in such a way that $\omega(t) \in \bigcap_{p=1}^{n} v_p(t)(1+\boldsymbol{\gamma})$ $\forall t \geq t_0$, which guarantees a particular consonance according to Lemma 7. In Section 5.1 we will see that $\omega(t)$ could make discrete leaps at resynchronization instants and remains constant otherwise in the proximity of the ideal rate 1. We mention explicitly that internal global rate is purely artificial, nevertheless, it allows us to reason about consonance by considering each clock separately, which provides many interesting insights and simplifies the analysis tremendously. All further considerations about internal clock rate synchronization rely on the definition of $\boldsymbol{\gamma}$-correctness upon rate intervals.

Definition 4 *Given consonance interval $\boldsymbol{\gamma}_p = [\gamma_p^-, 0, \gamma_p^+]$ and a clock $\mathcal{C}_p$ with its rate $v_p(t)$. A rate interval $\mathbf{R}_p$ is $\boldsymbol{\gamma}_p$-correct w.r.t. internal global rate $\omega(t)$ during a non-empty (real time) interval $\mathbf{T}$ iff*

$$\omega(t) \in v_p(t)\left(\mathrm{ref}(\mathbf{R}_p) + \boldsymbol{\gamma}_p\right) \qquad \forall t \in \mathbf{T}.$$

For an ensemble of clocks $\mathcal{C}_1, \ldots, \mathcal{C}_n$ the associated set of correct rate intervals $\boldsymbol{\mathcal{R}} = \{\mathbf{R}_1, \ldots, \mathbf{R}_n\}$ is $\boldsymbol{\gamma}$-correct during $\mathbf{T}$ iff each of them is $\boldsymbol{\gamma}$-correct w.r.t. internal global rate $\omega(t)$ during $\mathbf{T}$.

Clearly, $\boldsymbol{\gamma}$-correctness of a set of rate intervals implies $\boldsymbol{\gamma}$-consonance, but the converse is not necessarily true. In the same manner as rate intervals, associated consonance intervals can be deteriorated locally and transferred between nodes by keeping up their $\boldsymbol{\gamma}$-correctness, cf. Lemma 3 and 6. This is obvious, since internal global rate $\omega(t)$ takes over the role of the ideal rate 1 in the corresponding relations.

4 Clock Rate Algorithm

In this section we develop a generic algorithm CRA apt for both internal and external clock rate synchronization based on the building blocks from the preceding section.

With *rate resynchronization period* P_R each node executes the same algorithm. As already mentioned, the rounds are a product of a coexistent CSA with *state resynchronization period* P_S. To facilitate an easy implementation, P_R should be an integer multiple of P_S, since the engendered clock synchronization traffic from the CSA can be reused by the CRA accordingly, see Figure 3.

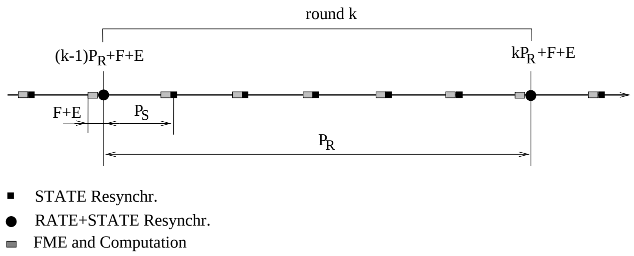

Figure 3: Rounds

Let us examine a particular rate round k, which starts at $(k-1)P_R + F + E$. The rates of the clocks are going to be adjusted by our algorithm at these points in time symbolized by circles in Figure 3. During a round, we need to carry out the protocol of Figure 2 in order to make relative rate measurements. More specifically, shortly before the beginning of round k, we initiate a *full message exchange* (FME) for the first messages, and near the end of the round for the second messages. These FMEs are already generated by the CSA, which adjusts the state of the clocks at the times symbolized by both squares and circles in Figure 3. Intervening state synchronizations are allowed to occur during relative rate measurements, since they can be easily taken into account by summing up the appropriate state adjustments.

To make our algorithm running properly, we have to set up large enough delays to account for the longest possible duration of any FME and any execution of algorithm CRA. In particular, delay F covers the worst case FME duration and E the worst case execution time.

Lemma 8 *Complying to assumptions (A1) ... (A8), any full message exchange is completed within $F = (1+2\rho_{\max})$ $(\pi_{\max}/(1-2\rho_{\max}) + \tau_{\max} + \lambda_{\max} + \Delta t_{\max} + \epsilon_{\max})$ logical seconds, and any single execution for the clock rate synchronization within $E = (1+2\rho_{\max})\eta_{\max}$ logical seconds when the participating nodes are non-faulty.*

Proof Sketch Imagine a slowest clock with rate $1 - 2\rho_{\max}$ that participates as latest in an FME and that experiences maximum delays $\tau_{\max}$, $\lambda_{\max}$ and $\Delta t_{\max} + \epsilon_{\max}$ for its broadcast, cf. assumption (A6), (A7) and (A8). Mapping the sum of them onto the fastest clock with rate $1 + 2\rho_{\max}$ that is also $\pi_{\max}$ ahead proves the first part. Similarly, mapping $\eta_{\max}$ onto the fastest clock provides the second part. □

Now we have prepared all parts and join them together in algorithm CRA. Periodically, at kP_R each node p initiates a broadcast of timestamped packet(s), which contain the local rate interval $\mathbf{R}_p$ that is correct throughout round k, the sum U_p of applied state adjustments during round k, and the sending timestamp T_p. Until $kP_R + F$ each node p receives packets from other nodes q that will be timestamped with $T_{p,q}$ upon reception and collected for further processing. Remembering the timestamps $T_q^{\prec}$ and $T_{p,q}^{\prec}$ of the former round enables us to carry out the protocol for relative rate measurement, cf. Figure 2. The quotient rate interval $\mathbf{Q}_{p,q}$ can be computed via Lemma 5 in a reduced version; note also that $\mathbf{Q}_{p,p}$ is not explicitly required because $\mathbf{R}_{p,p}$ is simply $\mathbf{R}_p$. Next the received local rate intervals $\mathbf{R}_q$ are transformed into remote rate intervals $\mathbf{R}_{p,q}$ by virtue of Lemma 6. An additional deterioration ensures a proper matching between the rate measurement period and the round endpoints, in order to make all $\mathbf{R}_{p,q}$ compatible with each other at resynchronization point $kP_R + F + E$.

Subsequently, the remote rate intervals $\mathbf{R}_{p,q}$ are fed into an interval-based *convergence function* $\mathcal{CV}_{\mathcal{F}}(\cdot)$ and/or *validation function* $\mathcal{VAL}_{\mathcal{F}}(\cdot)$ to get a new rate interval $\mathbf{R}'_p$. It should be both correct (external rate synchronization) and γ'-correct (internal rate synchronization) for a certain γ'. Sections 4.1 and 4.3 are devoted to these functions.

Here it remains to explain how the algorithm accomplishes the rate adjustment at $kP_{\mathrm{R}} + F + E$. Since $\mathbf{R}'_p$ is supposed to be correct, our best approximation for the clock rate is simply $1/\mathrm{ref}(\mathbf{R}'_p)$, hence we reset the current coupling factor S_p by $S_p\mathrm{ref}(\mathbf{R}'_p)$. This multiplicative rate change has to be accompanied by a new local rate interval, given as $\mathbf{R}'_p/\mathrm{ref}(\mathbf{R}'_p)$. Additionally, we check for $|S_p f_p - 1| \le \rho_p^{\max}$ according to assumption (A1) and (A3) to ensure a feasible rate adjustment, otherwise the clock is declared as faulty.

Algorithm CRA *Each node p performs:*

at kP_R:
initiate broadcast of packet(s) $\langle \mathbf{R}_p, U_p, T_p \rangle$

until $kP_R + F$:
receive packets $\langle \mathbf{R}_q, U_q, T_q \rangle$ *from nodes q*
timestamped by $T_{p,q}$

at $kP_R + F$:
compute the set of quotient rate intervals $\boldsymbol{\mathcal{Q}}_p$,
whereby $\mathbf{Q}_{p,q} \leftarrow [(T_q - T_q^{\prec} + U_q)/(T_{p,q} - T_{p,q}^{\prec} + U_p) \pm ((\sigma_p + \sigma_q)(T_q - T_q^{\prec} + U_q)/2 + \epsilon_{\max}/(T_{p,q} - T_{p,q}^{\prec} + U_p))]$
for $q \neq p$

compute the set of remote rate intervals $\boldsymbol{\mathcal{R}}_p$,
whereby $\mathbf{R}_{p,q} \leftarrow \mathbf{Q}_{p,q} \cdot \mathbf{R}_q + [0 \pm (\sigma_p + \sigma_q)(F + E)]$
for $q \neq p$ and $\mathbf{R}_{p,p} \leftarrow \mathbf{R}_p$

compute new rate interval $\mathbf{R}'_p$ *as*
application of $\mathcal{CV}_{\mathcal{F}}(\cdot)$ *and/or* $\mathcal{VAL}_{\mathcal{F}}(\cdot)$ *over* $\boldsymbol{\mathcal{R}}_p$

compute new local rate interval $\mathbf{R}_p$,
whereby $\mathbf{R}_p \leftarrow \mathrm{norm}(\mathbf{R}'_p) + [0 \pm \sigma_p P_R]$

at $kP_R + F + E$:
adjust clock rate by setting $S_p \leftarrow \mathrm{ref}(\mathbf{R}'_p) S_p$

For the initial case $k = 0$ at t_0, we begin with local rate interval $\mathbf{R}_p = [\rho_p/(1+\rho_p), 1, \rho_p/(1-\rho_p)]$ and coupling factor $S_p = 1/f_p$ as justified by Lemma 1. The results of analyzing algorithm CRA, in particular, the worst case consonance and run of drifts will be given in Theorem 1 and 2, respectively.

4.1 Convergence Function

We know that each correct remote rate interval $\mathbf{R}_{p,q}$ computed by node p contains the searched value of $1/v_p$, which is the anchor for rate adjustments. Informally, the crux is to use a convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ for computing a new interval $\mathbf{R}'_p$ out of these intervals that encloses $1/v_p$ more closely. However, in a real system faults may lead to erroneous intervals. An abstract fault model $\mathcal{F}$ has to be set up that restricts the variety of faults in terms of the provided rate/consonance intervals, see Section 4.2. A suitable convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ is based on a such an abstract fault model, thus the subscript $\mathcal{F}$. Examples for $\mathcal{CV}_{\mathcal{F}}(\cdot)$ range from plain intersection, over Marzullo's function in [Mar84], to the orthogonal-accuracy convergence function in [Schm97] that, for instance, provides the usual 1/3 tolerance against arbitrary faults.

For illustration, we draw a simple scenario that provides a good insight how the algorithm works. Assuming three nodes with stable clocks having rates $v_1 = 0.8$ Sec/sec, $v_2 = 0.9$ Sec/sec and $v_3 = 1.3$ Sec/sec, the occurring rate intervals $\mathbf{R}_{p,q}$ viewed from a global perspective could be as depicted in Figure 4. Since all of them are correct in our example, plain intersection is used as $\mathcal{CV}_{\mathcal{F}}(\cdot)$ to obtain the new rate intervals $\mathbf{R}'_p$ that are rather short due to averaging effects.

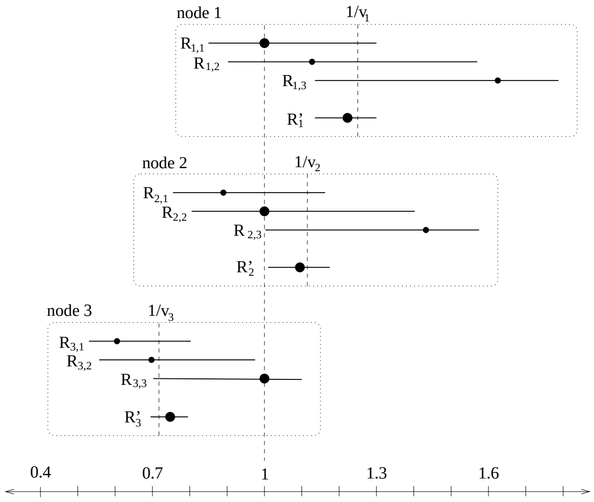

Figure 4: Global view of rate intervals

To understand internal rate synchronization, recall the notion of γ-correctness introduced in Definition 4. Suppose that each local rate interval belonging to a non-faulty clock is γ_0-correct w.r.t. $\omega(t)$ at the beginning of a particular round. During the round we have to deteriorate it in order to compensate for the oscillator stability accordingly. Choosing two different non-faulty nodes p and q, which have acquired a set of remote rate intervals $\{\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n}\}$ and $\{\mathbf{R}_{q,1}, \ldots, \mathbf{R}_{q,n}\}$, respectively, whose non-faulty ones are γ-correct at the end of this round for a suitable γ.

Applying the same convergence function at node p resp. q yields $\mathbf{R}'_p = \mathcal{CV}_{\mathcal{F}}(\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n})$ resp. $\mathbf{R}'_p = \mathcal{CV}_{\mathcal{F}}(\mathbf{R}_{q,1}, \ldots, \mathbf{R}_{q,n})$ that has to preserve/enhance consonance. More specifically, the clock rates have to be manipulated in such a way that the ensuing new local rate intervals become γ_0-consonant with $\gamma_0 \subset \gamma$. We emphasize that they cannot be safely asserted as γ_0-correct here, since it may be the case that the internal global rate $\omega(t)$ from the old round does not fit for the new consonance intervals. Defining a suitable new internal global rate resolves this deficiency and reassures γ_0-correctness, cf. Section 5.1.

Formally, in continuation of [SS97], [Schm97] and earlier [Schn87], the properties of a convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ for rate synchronization will be determined by a set of characteristic functions. Their arguments are consonance/rate intervals that specify the possible set of non-faulty remote rate intervals fed into $\mathcal{CV}_{\mathcal{F}}(\cdot)$. For the analysis of algorithm CRA it suffices to require that a particular convergence function is *translation invariant* as well as *weakly monotonic* in the sense of Definition 5 and complies to Definition 6.

Definition 5 *Given two sets* $\mathcal{I} = \{\mathbf{I}_1, \ldots, \mathbf{I}_n\}$ *and* $\mathcal{J} = \{\mathbf{J}_1, \ldots, \mathbf{J}_n\}$ *of* $n \geq 1$ *intervals. An interval-valued function* $\mathbf{f}(\cdot)$ *is called* translation invariant *iff* $\mathbf{f}(\mathbf{I}_1 + [0, y, 0], \ldots, \mathbf{I}_n + [0, y, 0]) = \mathbf{f}(\mathbf{I}_1, \ldots, \mathbf{I}_n) + [0, y, 0]$ *for any* $y \in \mathbb{R}$, *and* weakly monotonic *iff* $\mathbf{I}_i \subseteq \mathbf{J}_i$ *with* $\mathrm{ref}(\mathbf{I}_i) = \mathrm{ref}(\mathbf{J}_i)$ *for all* $1 \leq i \leq n$ *implies* $\mathbf{f}(\mathbf{I}_1, \ldots, \mathbf{I}_n) \subseteq \mathbf{f}(\mathbf{J}_1, \ldots, \mathbf{J}_n)$.

Definition 6 *Let* $\mathcal{R}_p = \{\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n}\}$ *and* $\mathcal{R}_q = \{\mathbf{R}_{q,1}, \ldots, \mathbf{R}_{q,n}\}$ *be two ordered sets of remote rate intervals in accordance with a given abstract fault model* $\mathcal{F}$*, where the non-faulty ones are subject to the following preconditions:*

[1] $\mathbf{R}_{p,i}$ *is correct and* $\mathbf{R}_{q,i}$ *is correct,*

[2] $\mathbf{R}_{p,i}$ *is* $\gamma_{p,i}$*-correct and* $\mathbf{R}_{q,i}$ *is* $\gamma_{q,i}$*-correct,*

[3] $\{\mathbf{R}_{p,i}, \mathbf{R}_{q,i}\}$ *is* γ^i*-correct,*

[4] $\mathcal{R}_p \cup \mathcal{R}_q$ *is* γ_H*-correct,*

[5] $\{\mathbf{R}_{p,i}, \mathbf{R}_{q,i}\}$ *is* γ_I*-consonant with* $||\gamma_I|| < ||\gamma_H||$,

[6] $\mathrm{align}(\mathbf{R}_{p,i}) \subseteq \mathbf{V}_{p,i}$*, and* $\mathrm{align}(\mathbf{R}_{q,i}) \subseteq \mathbf{V}_{q,i}$.

Provided that $\mathbf{R}'_p = \mathcal{CV}_{\mathcal{F}}(\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n})$ *and* $\mathbf{R}'_q = \mathcal{CV}_{\mathcal{F}}(\mathbf{R}_{q,1}, \ldots, \mathbf{R}_{q,n})$ *a generic convergence function* $\mathcal{CV}_{\mathcal{F}}(\cdot)$ *is characterized by*

(1) the correctness of both $\mathbf{R}'_p$ *and* $\mathbf{R}'_q$,

(2) a weakly monotonic consonance preservation function $\mathcal{CP}(\cdot)$ *iff* $\mathbf{R}'_p$ *is* $\mathcal{CP}(\gamma_{p,1} \ldots, \gamma_{p,n}; \gamma_H; \gamma_I; \ldots)$*-correct with* $||\mathcal{CP}(\gamma^1 \ldots, \gamma^n; \gamma_H; \gamma_I; \ldots)|| = \mathcal{O}(||\gamma_H||)$ *and analogous for* $\mathbf{R}'_q$,

(3) a weakly monotonic consonance enhancement function $\mathcal{CE}(\cdot)$ *iff* $\{\mathbf{R}'_p, \mathbf{R}'_q\}$ *is* γ'*-consonant satisfying* $||\gamma'|| = \mathcal{CE}(\gamma^1 \ldots, \gamma^n; \gamma_H; \gamma_I; \ldots) < ||\gamma_H||$,

(4) weakly monotonic drift preservation functions $\mathcal{DP}^{\pm}(\cdot)$ *iff* $\mathrm{align}(\mathbf{R}'_p) \subseteq [\mathcal{DP}^{-}(\mathbf{V}_{p,1}, \ldots, \mathbf{V}_{p,n}; \gamma_H; \gamma_I; \ldots), 0, \mathcal{DP}^{+}(\mathbf{V}_{p,1}, \ldots, \mathbf{V}_{p,n}; \gamma_H; \gamma_I; \ldots)]$ *and analogous for* $\mathrm{align}(\mathbf{R}'_q)$.

Let us briefly make of few informally remarks about the above definition. Obviously, item (1) requires that the computed rate intervals need to be correct for external clock rate synchronization. Item (2) helps to obtain the maximum amount of rate adjustments administered at the end of a round, since $\mathcal{CP}(\cdot)$ expresses how well the new rate intervals fit into the former round. Item (3) holds the key to maintain consonance of our clocks, because $\mathcal{CE}(\cdot)$ asserts a consonance property of the new rate intervals that allows to determine a new internal global rate. Finally, item (4) bounds the length of the new rate intervals via $\mathcal{DP}^{\pm}(\cdot)$ to account for the growth of clock drifts separated in a left and right length, which allows a strengthened analysis.

4.2 Abstract Fault Model

In order to cope with the variety of system faults, we ignore the actual cause (e.g. faulty component, excessive transmission delays, adversary) and pay attention to the resulting rate intervals. This explains the notion of an "abstract" fault model, cf. [SS97] and [Schm97]. Rate intervals can be faulty in terms of their reference points, their lengths, or missing altogether. This leads to a catalogue of fault types as shown in Table 1 from which a specific fault model $\mathcal{F}$ can be built. Furthermore, we rule out the possibility of *impersonating* other nodes or *jamming* the network.

fault type	remote rate interval(s)	node(s)
miss	missing	single
timing	non-correct and/or non-γ-correct due to faulty timing behavior	single
value	non-correct and/or non-γ-correct due to erroneous messages	single
crash	consistently missing	pair
omission	inconsistently missing	pair
restricted	consistently non-correct and/or non-γ-correct	pair
arbitrary	inconsistently non-correct and/or non-γ-correct	pair

Table 1: Classification of faulty remote rate intervals

4.3 Validation Function

For external rate synchronization we have to devise a mechanism to inject rate information from *primary nodes* $\mathcal{N}'$ in our system. Such nodes have their clocks disciplined towards a reference frequency obtained from atomic clocks or receivers for WWV, DCF77, or GPS, cf. [Dan97]. Obviously, their local rate intervals and consequently their contributing remote rate intervals are of short lengths compared to the ones from *non-primary nodes* $\mathcal{N}''$.

Just focusing on the set $\mathcal{X}_p$ of remote rate intervals from primary nodes appears to be a promising solution for external rate synchronization for a node p, since the new rate interval $\mathbf{R}'_p$ would be of short length too. However, there are basically two difficulties to exploit them: First, they might not be continuously available[2], which results in a toggling between internal and external rate synchronization. Second, an ordinary convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ is reluctant to obey them due to fault tolerance reasons.

One solution is the *clock validation* technique proposed in [Schm94], which compares the short but possibly faulty *reference* rate interval $\mathbf{X}'_p$ derived from nodes of $\mathcal{N}'$ with the longer but more reliable *validation* rate interval $\mathbf{R}'_p$ from nodes of $\mathcal{N}''$. If the reference rate interval is consistent with the validation rate interval, then clock $\mathcal{C}_p$ is adjusted via $\mathbf{X}'_p$ accomplishing external rate synchronization, otherwise, the algorithm proceeds with $\mathbf{R}'_p$ falling back to internal rate synchronization. Unfortunately, this straightforward technique works only in restricted fault models $\mathcal{F}$, in particular, certain arbitrary faults could separate the ensemble into groups of clocks with different drifts, cf. [FC97].

[2]We carried out a 2-month continuous experimental evaluation of the output of six different GPS receivers, which revealed a wide variety of failures, see [HS97] for details.

We are looking for new approaches pinned down by a validation function $\mathcal{VAL}_{\mathcal{F}}(\cdot)$ that integrates these two sets of remote rate intervals in a meaningful way. Moreover, inspired by [PR94] we are convinced that it is advantageous to consider the set of remote rate intervals from $\mathcal{N}'$ over the preceding h rounds as well, denoted by $\boldsymbol{\mathcal{X}}_p^{(-1)}, \ldots, \boldsymbol{\mathcal{X}}_p^{(-h)}$, which can be maintained by virtue of Lemma 3. A validation function in conjunction with a suitable convergence function gets applied on all these sets to compute

$$\mathbf{R}_p' = \mathcal{VAL}_{\mathcal{F}}\left(\boldsymbol{\mathcal{R}}_p; \boldsymbol{\mathcal{X}}_p; \boldsymbol{\mathcal{X}}_p^{(-1)}; \ldots; \boldsymbol{\mathcal{X}}_p^{(-h)}\right).$$

5 Analysis

The analysis covers roughly the dissemination of rate and consonance intervals within a single round, the maximal rate adjustments, the worst case consonance and drifts of clocks. For a more comprehensive analysis consult [Scho96].

5.1 Interval Dissemination

First, we have to agree upon the real time when a particular round starts, since clocks do not resynchronize simultaneously. Let $t_p^{(k)}$ be the real time when a non-faulty node p switches to round $k \geq 0$. For the purpose of analysis, rate/consonance intervals are only meaningful when they refer to a common point of time. The beginning of round k denoted as $t^{(k)}$ is determined by the latest resynchronization time $t_p^{(k)}$ among non-faulty nodes $p \in \mathcal{N}^{(k)}$, thus

$$t^{(k)} = \max\{t_p^{(k)} \mid p \in \mathcal{N}^{(k)}\}.$$

Since $\pi_{\max}$ is assumed to be small we can justify that $v_p(t^{(k)}) = v_p(t_p^{(k)}) + \mathcal{O}(\sigma_p \pi_{\max})$, which hands over to the rate/consonance intervals.

Basically, the analysis of our algorithm is built on an induction proof over the number of rounds, so we are interested how the rate/consonance intervals evolve during a round. We claim that there exists a set $\boldsymbol{\mathcal{R}}^{(k)} = \{\mathbf{R}_p(t^{(k)}) \mid p \in \mathcal{N}^{(k)}\}$ of local rate intervals at the beginning $t^{(k)}$, where $\mathbf{R}_p(t^{(k)}) + [0 \pm \mathcal{O}(\sigma_p \pi_{\max})]$ is correct and $\mathbf{R}_p(t^{(k)})$ is $(\boldsymbol{\gamma}^{(k)} + [0 \pm \mathcal{O}(\sigma_p \pi_{\max})])$-consonant. Recalling Definition 3 and 4, we can make each rate interval $(\boldsymbol{\gamma}^{(k)} + [0 \pm \mathcal{O}(\sigma_p \pi_{\max})])$-correct if we define internal global rate as follows: $\omega(t) = \omega^{(k)} = \omega(t^{(k)})$ for $t^{(k)} \leq t < t^{(k+1)}$ with $\omega(t^{(k)})$ taken arbitrary from

$$\bigcap_{p \in \mathcal{N}^{(k)}} v_p(t^{(k)}) \left(\mathrm{ref}(\mathbf{R}_p(t^{(k)})) + \boldsymbol{\gamma}^{(k)} + [0 \pm \mathcal{O}(\sigma_p \pi_{\max})]\right).$$

Lemma 9 *Given algorithm CRA under assumptions (A1) ... (A8). If $\mathbf{R}_q$ resp. $\mathbf{R}_p$ is a correct local rate interval during round k at remote node q resp. local node $p \neq q$, then the computed remote rate interval $\mathbf{R}_{p,q}$ has the following properties at the end of round k:*

(1) $\mathrm{ref}(\mathbf{R}_{p,q}) \in [1 \pm (||\mathbf{R}_p + \mathbf{R}_q|| + \frac{\sigma_p + \sigma_q}{2} P_R + (2B + \epsilon_{\max} + 4(F+E)\rho_{\max})/P_R)] + [0 \pm \mathcal{O}(G_{p,q})]$

(2) $\mathrm{align}(\mathbf{R}_{p,q}) \subseteq \mathrm{align}(\mathbf{R}_q) + [0 \pm (\frac{\sigma_p + \sigma_q}{2}(P_R + 2(F+E) + B) + \epsilon_{\max}/(P_R - B - \epsilon_{\max} - 4(F+E)\rho_{\max}))] + [0 \pm \mathcal{O}(G_{p,q} + \sigma_{\max}\pi_{\max})]$

with maximum broadcast delay $B = (1 + 2\rho_{\max})(\tau_{\max} + \lambda_{\max})$ and error term $G_{p,q} = (||\mathbf{R}_p + \mathbf{R}_q|| + \sigma_{\max} P_R + (B + \epsilon_{\max} + (F+E)\rho_{\max})/P_R)^2$.

Proof Sketch Since the computation of $\mathbf{R}_{p,q}$ involves the multiplication of $\mathbf{Q}_{p,q}$ according to Lemma 6, our first step is to obtain properties on the latter. Hence we want to find bounds on the duration $\Delta T_q = T_q - T_q^{\prec}$ and $\Delta T_{p,q} = T_{p,q} - T_{p,q}^{\prec}$ under the assumption of a transparent clock state synchronization. Directly from the algorithm and consulting the proof of Lemma 8 with assumption (A6) and (A7) we get

$$|\Delta T_q - P_R| \leq B,$$

where B is calculated as above. With the help of Lemma 4 and the correctness of $\mathbf{R}_q$ we carry over bounds on ΔT_q to the real time counterpart $\Delta t_q = t_q - t_q^{\prec}$ and further to $\Delta t_{p,q} = t_{p,q} - t_{p,q}^{\prec}$ via assumption (A8). Mapping them onto clock $\mathcal{C}_p$ with clock rates induced by the correct $\mathbf{R}_p$ leads to

$$\begin{aligned}|\Delta T_{p,q} - P_R| \quad \leq \quad & P_R\left(||\mathbf{R}_p|| + ||\mathbf{R}_q|| + \frac{\sigma_p + \sigma_q}{2} P_R\right) + \\ & B + \epsilon_{\max} + 4(F+E)\rho_{\max} + \mathcal{O}(G_{p,q})P_R,\end{aligned}$$

where $G_{p,q}$ is calculated as above. From Lemma 5 we know that $\Delta T_q / \Delta T_{p,q} = \mathrm{ref}(\mathbf{Q}_{p,q})$ which equals $\mathrm{ref}(\mathbf{R}_{p,q})$, and since any additional deteriorations have no effect on it, we able able to show item (1) of our lemma.

For item (2) we compute an upper bound on $||\mathbf{Q}_{p,q}||$ by plugging in the bounds for ΔT_q and $\Delta T_{p,q}$ into the formula of Lemma 5. Afterwards we apply the simple relation $\mathrm{align}(\mathbf{Q}_{p,q} \cdot \mathbf{R}_q) \subseteq \mathrm{align}(\mathbf{Q}_{p,q}) + \mathrm{right}(\mathbf{Q}_{p,q})\mathrm{align}(\mathbf{R}_q)$ and account for the additional deterioration $[0 \pm (\sigma_p + \sigma_p)(F+E)]$ to bridge the gap between the endpoints of the measurement period and the corresponding round. □

Lemma 10 *Given algorithm CRA under assumptions (A1) ... (A8). Let $\mathcal{N}^{(k)}$ be the set of non-faulty nodes during round k. If the set of local rate intervals $\boldsymbol{\mathcal{R}}^{(k)} = \{\mathbf{R}_p \mid p \in \mathcal{N}^{(k)}\}$ is correct during round k and $\boldsymbol{\gamma}^{(k)}$-correct w.r.t. internal global rate $\omega^{(k)}$ at the beginning of round k, then the computed remote rate intervals $\mathbf{R}_{p,q}$ for $p, q \in \mathcal{N}^{(k)}$ have the following properties at the end of the round k:*

(1) Any remote rate interval $\mathbf{R}_{p,q}$ is $\boldsymbol{\gamma}_{p,q}$-correct with

$\boldsymbol{\gamma}_{p,q} \subseteq \boldsymbol{\gamma}^{(k)} + [0 \pm (\frac{\sigma_p + \sigma_q}{2}(P_R + 2(F+E) + B) + \epsilon_{\max}/(P_R - B - \epsilon_{\max} - 4(F+E)\rho_{\max}))] + [0 \pm \mathcal{O}(G_{\max} + \sigma_{\max}\pi_{\max})]$

for $p \neq q$, and $\mathbf{R}_{p,p}$ is $\boldsymbol{\gamma}_{p,p}$-correct with

$$\boldsymbol{\gamma}_{p,p} \subseteq \boldsymbol{\gamma}^{(k)} + [0 \pm \sigma_p P_R] + [0 \pm \mathcal{O}(G_{\max} + \sigma_{\max}\pi_{\max})].$$

(2) The set of remote rate intervals $\boldsymbol{\mathcal{R}}^q = \{\mathbf{R}_{p,q} \mid p \in \mathcal{N}^{(k)}\}$ at receiving nodes $p \in \mathcal{N}^{(k)}$ from a broadcasting node q is $\boldsymbol{\gamma}^q$-correct with

$\boldsymbol{\gamma}^q \subseteq \boldsymbol{\gamma}^{(k)} + [0 \pm (\frac{\sigma_{\max} + \sigma_q}{2}(P_R + 2(F+E) + B) + \epsilon_{\max}/(P_R - B - \epsilon_{\max} - 4(F+E)\rho_{\max}))] + [0 \pm \mathcal{O}(G_{\max} + \sigma_{\max}\pi_{\max})]$,

and $\boldsymbol{\gamma}_I$-consonant with

$\boldsymbol{\gamma}_I \subseteq [0 \pm (\sigma_{\max}(P_R + 2(F+E) + \frac{3}{2}B) + \epsilon_{\max}/(P_R - B - \epsilon_{\max} - 4(F+E)\rho_{\max}))] + [0 \pm \mathcal{O}(G_{\max} + \sigma_{\max}(\pi_{\max} + B\rho_{\max}))]$.

(3) The set of all remote rate intervals $\mathcal{R} = \{\mathbf{R}_{p,q} \mid p,q \in \mathcal{N}^{(k)}\}$ at non-faulty nodes is γ_H-correct with

$$\gamma_H \subseteq \gamma^{(k)} + [0 \pm (\sigma_{\max}(P_R + 2(F+E) + B) + \epsilon_{\max}/(P_R - B - \epsilon_{\max} - 4(F+E)\rho_{\max}))] + [0 \pm \mathcal{O}(G_{\max} + \sigma_{\max}\pi_{\max})].$$

Error term $G_{\max}$ is given as $(||\mathbf{R}_{\max}|| + \sigma_{\max}P_R + (B+\epsilon_{\max} + (F+E)\rho_{\max})/P_R)^2$, where $\mathbf{R}_{\max}$ is the largest correct local rate interval.

Proof Sketch The first part of item (1) of our lemma is analogous to item (2) of Lemma 9, but instead of rate intervals the associated consonance intervals are analyzed, hence it is sufficient to make two substitutions, namely $\mathrm{align}(\mathbf{R}_{p,q})$ becomes to $\gamma_{p,q}$ and $\mathrm{align}(\mathbf{R}_q)$ to $\gamma^{(k)}$. The second part is trivial, since $\mathbf{R}_{p,p} = \mathbf{R}_p$.

The first part of item (2) and item (3) and are essentially majorizations over the appropriate non-faulty nodes.

To prove the second part of item (2) we operate with a consonance interval $[0 \pm \sigma_q B]$ when the last message leaves the broadcasting node q. Then we proceed as in item (1) and apply pertinent majorizations. □

5.2 Rate Adjustment

We investigate the amounts of rate adjustments for internal rate synchronization administered at the end of each round by changing instantaneously the coupling factor S_p.

Lemma 11 *Given algorithm CRA under assumptions (A1) ... (A8) using convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ subject to Definition 5 and 6. Let $\mathcal{N}^{(k)}$ be the set of non-faulty nodes during round k. If the set of local rate intervals $\mathcal{R}^{(k)} = \{\mathbf{R}_p \mid p \in \mathcal{N}^{(k)}\}$ is correct during round k and $\gamma^{(k)}$-correct w.r.t. internal global rate $\omega^{(k)}$ at the beginning of round k, then the rate adjustment at the end of the round k, expressed as the ratio between the new and old coupling factor, is limited by*

$$\begin{aligned} S_p^{(k+1)}/S_p^{(k)} \quad \in \quad & 1 + \gamma^{(k)} + [0 \pm \sigma_p P_R] + \\ & \mathrm{swap}(\mathcal{CP}(\gamma_{p,1} \ldots, \gamma_{p,n}; \gamma_H; \gamma_I; \ldots)) + \\ & [0 \pm \mathcal{O}(G_{\max} + \sigma_{\max}\pi_{\max})] \end{aligned}$$

for node $p \in \mathcal{N}^{(k)}$, whereby $\mathcal{CP}(\gamma_{p,1} \ldots, \gamma_{p,n}; \gamma_H; \gamma_I; \ldots)$ is the weakly monotonic consonance preservation function of $\mathcal{CV}_{\mathcal{F}}(\cdot)$ supplied with $\gamma_{p,q}$, γ_H and γ_I from Lemma 10.

Proof Sketch Since $S_p^{(k+1)} = S_p^{(k)}\mathrm{ref}(\mathbf{R}'_p)$ and knowing that $\mathbf{R}_{p,p}$ is $\gamma_{p,p}$-correct from Lemma 10 and $\mathbf{R}'_p$ is $\mathcal{CP}(\gamma_{p,1} \ldots, \gamma_{p,n}; \gamma_H; \gamma_I; \ldots)$-correct w.r.t. the same internal global rate enables us to find the desired quantity. □

5.3 Consonance

The following theorem expresses the maximal lengths of the ensuing consonance intervals at the beginning of each round, whereby the conversion to the usual consonance can be done via Lemma 7.

Theorem 1 *Given algorithm CRA under assumptions (A1) ... (A8) using convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ subject to Definition 5 and 6. Let γ^* be a solution of equation*

$$||\gamma^*|| = \mathcal{CE}(\gamma^1 \ldots, \gamma^n; \gamma_H; \gamma_I; \ldots),$$

whereby $\mathcal{CE}(\gamma^1 \ldots, \gamma^n; \gamma_H; \gamma_I; \ldots)$ is the weakly monotonic consonance enhancement function of $\mathcal{CV}_{\mathcal{F}}(\cdot)$ supplied with γ^q, γ_H and γ_I from Lemma 10. If each local rate interval $\mathbf{R}_p$ of a non-faulty node p is correct up to the beginning $t^{(k)}$ of round $k \geq 1$ with $\mathbf{R}_p \subseteq \mathbf{R}_{\max}$, and initially $\gamma^{(0)}$-correct with $\gamma^{(0)} \subseteq \gamma^*$ at $t^{(0)}$, then $\mathbf{R}_p$ is $(\gamma^{(k)} + [0 \pm \mathcal{O}(G_{\max} + \sigma_{\max}\pi_{\max})])$-correct at $t^{(k)}$ with $\gamma^{(k)} \subseteq \gamma^*$.

Proof Sketch We establish this result by an induction proof on round k, whereby the initial case $k = 0$ is already implied by the above assumption.

Suppose the set $\{\mathbf{R}_p^{(k)} \mid p \in \mathcal{N}^{(k)}\}$ of local rate intervals is $\gamma^{(k)}$-correct w.r.t. $\omega^{(k)}$ at $t^{(k)}$. When round k ends at $t^{(k+1)}$, each node has computed its new rate interval $\mathbf{R}_p'^{(k)}$, whereby the set $\{\mathbf{R}_p'^{(k)} \mid p \in \mathcal{N}^{(k)}\}$ is $\gamma'^{(k)}$-consonant according to Definition 6. The intervals $\gamma^1 \ldots, \gamma^n$, γ_H and γ_I taken over as bounds from Lemma 10 plugged into the weakly monotonic $\mathcal{CE}(\cdot)$ determine a maximum on $||\gamma'^{(k)}||$, hence the right hand side of relation $||\gamma'^{(k)}|| \leq \mathcal{CE}(\gamma^1 \ldots, \gamma^n; \gamma_H; \gamma_I; \ldots)$ is primarily dependent on $\gamma^{(k)}$.

Establishing a new internal global rate $\omega^{(k+1)}$ as given in Section 5.1 warrants that each $\mathbf{R}_p'^{(k)}$ is $\gamma'^{(k)}$-correct. After the rate adjustment took place at $t^{(k+1)}$, the new local rate interval $\mathbf{R}_p^{(k+1)} = \mathrm{norm}(\mathbf{R}_p'^{(k)})$ is $\gamma^{(k+1)}$-correct, whereby $||\gamma^{(k+1)}|| = ||\gamma'^{(k)}|| + \mathcal{O}(G_{\max} + \sigma_{\max}\pi_{\max})$ due to Lemma 11 and our simplifications $||\mathcal{CP}(\gamma^1 \ldots, \gamma^n; \gamma_H; \gamma_I; \ldots)|| = \mathcal{O}(||\gamma_H||)$ and $\mathcal{CE}(\gamma^1 \ldots, \gamma^n; \gamma_H; \gamma_I; \ldots) < ||\gamma_H||$ as part of Definition 6.

To make our induction step working, we require that $||\gamma^{(k+1)}|| \leq ||\gamma^{(k)}||$, which leads to the equation of our lemma with the unknown γ^*. If it exists, the weak monotonicity of $\mathcal{CV}_{\mathcal{F}}(\cdot)$ makes it applicable to any consonance interval enclosed by γ^*. □

5.4 Drift

The following theorem expresses recursively the lengths of the local rate intervals at the beginning of each round, whereby the conversion to the usual clock drift δ_p can be done via Lemma 1. However, it is overly conservative since we did not consider the fact that a maximal rate adjustment cannot take place more than once in consecutive rounds, cf. [Schm97]. An improvement can be achieved by using "conditional" drift preservation functions $\mathcal{DP}^{\pm}(\cdot)$ that also consider the length of the interval from which internal global rate is drawn, since it acts as a record of earlier applied rate adjustments.

Theorem 2 *Given algorithm CRA under assumptions (A1) ... (A8) using convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ subject to Definition 5 and 6. Let all local rate intervals $\mathbf{R}_p$ correct at $t^{(0)}$ with $\mathrm{align}(\mathbf{R}_p) \subseteq \mathbf{V}_p^{(0)}$. If node p is non-faulty up to the beginning $t^{(k)}$ of round $k \geq 1$, then $\mathbf{R}_p$ is correct at $t^{(k)}$ and satisfies $\mathrm{align}(\mathbf{R}_q) \subseteq \mathbf{V}_p^{(k)} + [0 \pm \mathcal{O}(G_{\max} + \sigma_{\max}\pi_{\max})]$, whereby $\mathbf{V}_p^{(k)} = [\mathcal{DP}^-(\mathbf{V}_{p,1}^{(k)}, \ldots, \mathbf{V}_{p,n}^{(k)}; \gamma_H; \gamma_I; \ldots), 0, \mathcal{DP}^+(\mathbf{V}_{p,1}^{(k)}, \ldots, \mathbf{V}_{p,n}^{(k)}; \gamma_H; \gamma_I; \ldots)]$ with*

$$\begin{aligned} \mathbf{V}_{p,q}^{(k)} \quad = \quad & \mathbf{V}_q^{(k-1)} + [0 \pm ((\sigma_p + \sigma_q)(P_R + 2(F+E) + B)/2 + \\ & \epsilon_{\max}/(P_R - B - \epsilon_{\max} - 4(F+E)\rho_{\max}))] \end{aligned}$$

Proof Sketch By induction on round k and stringing together the rate intervals bounds from item (1) of Lemma 9 with them from item (4) of Definition 6. □

Conclusion and Future Work

We have crafted a framework for algorithms that synchronize the rates of clocks in a distributed system. Its interval-based paradigm is surprisingly similar in many aspects to ours for clock state synchronization presented in [SS97]. In fact, the same convergence/validation functions can be used to operate on intervals that captured synchronization quantities like clock rates or real time. These intervals in turn need to be sufficiently enlarged to account for uncertainties arising from the communication subsystem or the clocks. For the latter we have discovered that their stabilities have great importance for (internal) clock synchronization instead of their conservative maximum drift.

Much work remains to be done. In particular, we are about to investigate the integration of clock rate and state synchronization algorithms in more details including the initialization/joining problem, and to analyze promising convergence/validation functions along with abstract fault models. Moreover, we are interested in optimality results for clock rate synchronization.

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