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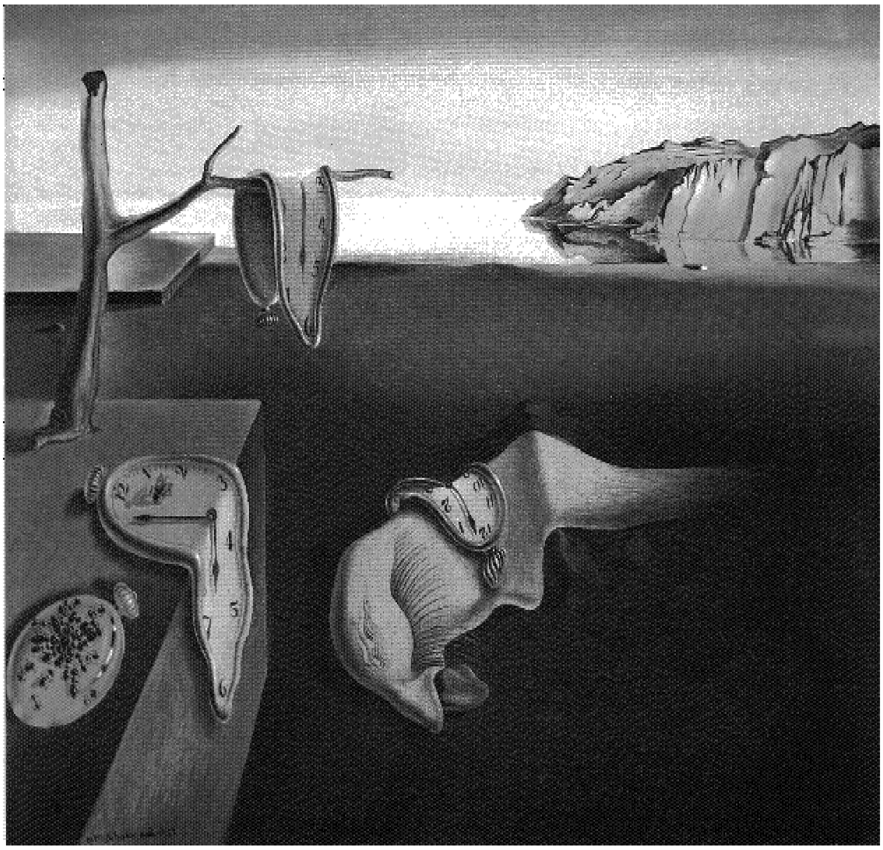

Salvador Dali, "Die Beständigkeit der Erinnerung"

Understanding Interval-based Clock Rate Synchronization Algorithms *

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Abstract

In this paper we promote rate synchronization for clocks in fault-tolerant distributed systems and develop the underpinnings for a clock rate algorithm. Two parameters are used for characterization, namely drift for external and consonance for internal clock rate synchronization. A clock rate algorithm is similar to a conventional clock state algorithm, but instead of depending on their maximum oscillator drifts, their stability is exploited. We present a comprehensive system model, work out the concepts for clock rate synchronization and give a general analysis by using a suitable interval paradigm.

Keywords: distributed systems, clock state algorithm, clock rate algorithm, accuracy, precision, drift, consonance, stability, rate intervals, convergence functions, clock validation.

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Contents

1 Introduction

The concept of time allows us to reason about events and durations. For the purpose of quantification, designated clocks attach (real) number to them, so we are able to tell when an event occurs and how long a duration takes. This mapping in turn characterizes clocks in two ways: An event draws a particular clock state, whose deviations towards a reference clock are known as accuracy. A duration, measured as the difference between two corresponding clock states, enables us to determine a particular clock rate when compared against a reference clock, whereby deviations are known as drift. Unlike the states of clocks, however, the rates are not directly observable.

In case of an ensemble of clocks, a common event could lead to distinct clock states, if they are not synchronized in terms of state. A clock state synchronization algorithm is responsible to guarantee a maximum state difference among them, called precision. Similarly, a common duration could reveal distinct clock rates, and a separate clock rate synchronization algorithm has to take care for a maximum rate difference, called consonance. Note that a small precision compels a small consonance, but the reverse need not to be true. One goal of this paper is to shed more light on the relationship between clock state and rate synchronization.

A vast material on clock state synchronization has been developed in the last 2 decades, nevertheless, many algorithms are content with rate synchronized clocks, cf. [Lis93]. Distributed applications in computer science based on time-outs (e.g. lifetime of Kerberos tickets) or round-trips (e.g. NTP protocol) serve as representative examples. Still, state synchronized clocks are essential for many applications (e.g. real-time systems), but tight and robust synchronizations are expensive in terms in components and bandwidth. Clocks having their rates synchronized beyond the usual manufacturer's drift specifications, offers the possibility to achieve a higher performance, simplifies initial clock synchronization or facilitates a fault detection mechanism.

Not much research has been conducted towards the problem of clock rate synchronization. The most important contribution is the pioneering work of [Mar84], presenting an algorithm based on non-accelerating clocks with a round-trip method for measuring clock rates. Intervals were used to capture the relevant information for the fault-tolerant algorithm. Our goal is to extend his ideas, resulting in the development of an algorithm for clock rate synchronization, that deals with both drift and consonance embedded in a realistic system model. Furthermore, we work out the similarities to the well-known problem of clock state synchronization, in order to benefit from it's rich collection of theoretical results and approaches, cf. [Sch87] or [SWL90].

The paper is organized as follows: Section 2 introduces formally the clock rate synchronization problem and stipulates our system model concerning clocks and processors along with their means of communication. After a comprehensive preparatory work on notations and building blocks for rate synchronization in Section 3, the following section develops the theory of an algorithm both for external and internal rate synchronization. Section 5 provides the analysis of drift and consonance in a generic way, since no particular fault model is applied. Concluding remarks and future research issues close the paper.

2 System Modeling

Each *node* p in the distributed system hosts a local *clock* $\mathcal{C}_p$ regarded as an entity that reads *clock state* T at the non-directly observable *real-time* t in some meaningful Newtonian frame. In mathematical parlance, a clock can be described by a piecewise continuous function $C_p : t \mapsto T$ by neglecting granularity issues. Ideally, C_p should be the identity function, but reality forces us to be content with an approximation resulting in partial synchronization.

2.1 Synchronizing Clocks in State and Rate

Usually, we characterize clock synchronization in terms of *clock states* by two parameters: The maximum deviation between corresponding clock states and real-times on a single clock is called *accuracy* α, and the maximum clock state deviation between two different clocks in the distributed system at simultaneous real-times is called *precision* π. Maintaining accuracy resp. precision of an ensemble of clocks refers to the problem of *external* resp. *internal clock state synchronization*.

Definition 1 (Clock Accuracy and Precision) *Let* $\mathbf{T}$ *be a non-empty real-time period. A clock* $\mathcal{C}_p$ *has accuracy* α_p *during* $\mathbf{T}$ *iff* $|C_p(t) - t| \leq \alpha_p \ \forall t \in \mathbf{T}$. *Any two different clocks* $\mathcal{C}_p$ *and* $\mathcal{C}_q$ *have precision* π *during* $\mathbf{T}$ *iff* $|C_p(t) - C_q(t)| \leq \pi \ \forall t \in \mathbf{T}$.

However, we can view clock synchronization in terms of *clock rates* as well. Denoting the derivative $dC_p(t)/dt$ by the instantaneous clock rate $v_p(t)$, where $C_p(t)$ is differentiable, we obtain another two parameters for characterization measured in Sec/sec. The maximum deviation between the clock rate and the ideal rate 1 is denoted by *drift*, and the maximum clock rate deviation between two different clocks in the distributed system at simultaneous real times is called *consonance*. Maintaining drift resp. consonance of an ensemble of clocks refers to the problem of *external* resp. *internal clock rate synchronization*.

Definition 2 (Clock Drift and Consonance) *Let* $\mathbf{T}$ *be a non-empty real-time period. A clock* $\mathcal{C}_p$ *has drift* δ_p *during* $\mathbf{T}$ *iff* $|v_p(t) - 1| \leq \delta_p \ \forall t \in \mathbf{T}$, *where* $C_p(t)$ *is differentiable. Any two different clocks* $\mathcal{C}_p$ *and* $\mathcal{C}_q$ *have consonance* γ *during* $\mathbf{T}$ *iff* $|v_p(t) - v_q(t)| \leq \gamma \ \forall t \in \mathbf{T}$, *where* $C_p(t)$ *and* $C_q(t)$ *are differentiable.*

Figure 1 illustrates the components involved in steering a local clock. On the one hand, a *clock state algorithm* (CSA) is in charge of accuracy/precision, and on the other hand, a *clock rate algorithm* (CRA) targets drift/consonance. The benefits of a CRA are twofold: Besides achieving a better measurement for time durations it can support a coexisting CSA by reducing the accuracy/precision deterioration of freely running clocks during consecutive state resynchronization instants. This effect open up the possibility to save communication bandwidth or to achieve a tighter clock state synchronization.

2.2 Local Clock

A local clock $\mathcal{C}_p$ is attached to a local *oscillator* $\mathcal{O}_p$ in order to keep pace with the progress of time. The oscillator indicates a passage of time with periodic ticks of nominal *frequency* f_p given in $ticks/sec$. The manufacturer specifies a maximum *oscillator drift* ρ_p in *ppm*, hence the instantaneous oscillator frequency $f_p(t)$ is bounded by $f_p(1 - \rho_p) \leq f_p(t) \leq f_p(1 + \rho_p)$. The distinction between oscillator and clock is the hook to introduce clock rate synchronization. In the absence of a CRA, the oscillator is directly coupled to the clock in the sense that

$$v_p(t) = S_p f_p(t), \tag{1}$$

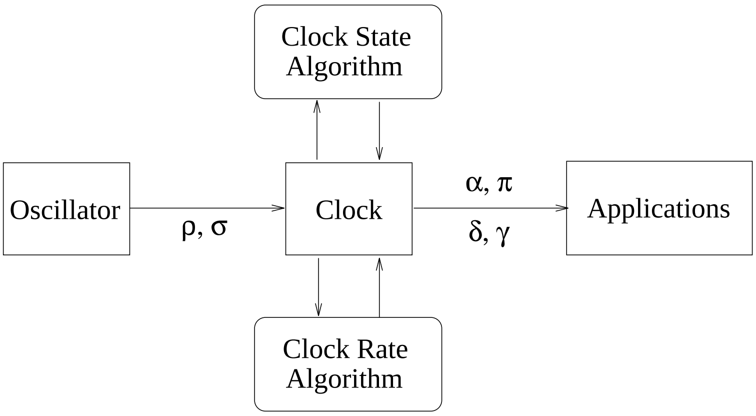

Figure 1: Clock Steering

where *coupling factor* S_p is the constantly $1/f_p$ $Sec/tick$. In such an arrangement clock drift δ_p becomes the maximum oscillator drift ρ_p, which is determined by the physics and environment of the oscillator.

It is vital to understand that a CRA tries to break up this rigid oscillator-clock coupling, by getting a handle on S_p. In other words, even if the oscillator frequency $f_p(t)$ changes, the factor S_p should be influenced in such a way, that the clock rate $v_p(t)$ remains to be approximately constant. Obviously, the realization of a local clock needs to provide a means to set S_p like the CSU from [KO87] or our UTCSU, cf. [SSHL97]. The feasible range of coupling factor S_p is given by $1/f_p(1 \pm \rho_p)$, through which the rate $v_p(t)$ of a non-faulty clock satisfies

$$|v_p(t) - 1| \leq 2\rho_p. \tag{2}$$

Clock rate synchronization rests on the fact, that the frequency $f_p(t)$ of oscillator $\mathcal{O}_p$ does not alter too quick, otherwise a permanent adjustment of factor $S_p(t)$ would become necessary to control clock $\mathcal{C}_p$. Depending solely on the specified maximum oscillator drift ρ_p, the oscillator frequency could make instantaneous leaps up to $2f_p\rho_p$, but experiments show, that oscillators keep up their frequency to some extent. For a CRA it is essential to make proper assumptions about the oscillator frequency, cf. [NIST90] or [Tro94]. In particular, we should consider a dynamic characterization, which should include short-term stochastic frequency changes stemming from various noise sources, shock, or radiation, and long-term deterministic changes from aging, temperature or humidity.

Suppose oscillator $\mathcal{O}_p$ has instantaneous frequency $f_p(t_1)$ at real-time t_1 and $f_p(t_2)$ at $t_2 \geq t_1$. We stipulate that the ratio $\frac{f_p(t_2)}{f_p(t_1)}$ is linearly bounded by $1 \pm \sigma_p(t_2 - t_1)$. Parameter σ_p is called the *oscillator stability* measured in ppm/sec for treating the long-term deterministic effects. However, this characterization[1] is only meaningful for certain durations $0 \leq t_2 - t_1 < \infty$. For very large ones, the stability term $\sigma_p(t_2 - t_1)$ would become too pessimistic, since if $t_2 - t_1$ becomes $1/\sigma_p$ the clock could possibly stop or run twice its nominal rate. In fact, the given specified maximum oscillator drift ρ_p prevents the frequency ratio on the long run from exceeding $1 \pm 2\rho_p$. For very small durations, say a few oscillator periods $1/f_p$, glitches from short-term stochastic effects could violate the bounds, but we regard them as non-accumulating. If fact, measurements on oscillators confirm

[1] A more advanced way for characterizing the oscillator would be $\frac{f_p(t_2)}{f_p(t_1)} \in [1 \pm (\nu_p + \sigma_p(t_2 - t_1))]$.

this hypothesis, but an increased variance arises. Let us summarize our oscillator assumptions, which hold for proper parameter settings.

Assumption 1 (Oscillator Drift and Stability) *Each non-faulty node p hosts an oscillator* $\mathcal{O}_p$ *with instantaneous frequency* $f_p(t)$ *subject to two conditions. The drift condition*

$$\left| \frac{f_p(t)}{f_p} - 1 \right| \leq \rho_p \qquad \forall\ t \geq t_0 \tag{3}$$

bounds the instantaneous frequency, where f_p *is the nominal oscillator frequency and* ρ_p *the maximum oscillator drift. The stability condition*

$$\left| \frac{f_p(t_2)}{f_p(t_1)} - 1 \right| \leq \sigma_p(t_2 - t_1) \qquad \forall\ t_2 \geq t_1 \geq t_0 \tag{4}$$

bounds the variation of the instantaneous frequencies during $t_2 - t_1$, *where* σ_p *is the oscillator stability.*

In the special case of $\sigma_p = 0$ we say the oscillator/clock is *stable*, thus $f_p(t) = \text{const}$. Moreover, we define $\sigma_{\max} = max_{i=1}^{n} \sigma_i$ as the uniform bound on the oscillator stability, and $\rho_{\max} = max_{i=1}^{n} \rho_i$ as the uniform maximum oscillator drift. Note that expressions like $\sigma_p(t_2 - t_1)$ are regarded to be in the magnitude of ρ_p.

It is important to realize, that a CRA has no means to affect the oscillator stability. A free running clock $\mathcal{C}_p$ inherits this property directly for its rate, since the common coupling factor S_p falls out of the ratio, formally

$$\left| \frac{v_p(t_2)}{v_p(t_1)} - 1 \right| \leq \sigma_p(t_2 - t_1). \tag{5}$$

Apart from making assumptions on the clock rates, we additionally presume a weak clock state synchronization as a result of the coexisting CSA. In particular, only a global precision $\Pi_{\max}$ is required to have a measure on the degree of simultaneity in our system, relinquishing a certain global accuracy of the clocks. The dependence among these algorithms is rather loose, since the CSA offers its precision and the CRA exports tight bounds on the clock drifts.

Assumption 2 (Global Precision) *The clocks in our system are synchronized by an underlying clock state algorithm, where any non-faulty pair of clocks* $\mathcal{C}_p$ *and* $\mathcal{C}_q$ *satisfies the precision condition*

$$|C_p(t) - C_q(t)| \leq \Pi_{\max} \qquad \forall\ t \geq t_0. \tag{6}$$

2.3 Processor

Besides a local clock each node is equipped with a *processor* that executes the CRA. As it will turn out, all computations are essentially periodic and require floating-point arithmetic with sufficient resolution. In terms of the execution speed and organization of the processor, we make the following assumption.

Assumption 3 (Computation Times) *A single computation required for clock rate synchronization at a non-faulty node is completed within* $\eta_{\max}$ *seconds.*

2.4 Communication Subsystem

Nodes communicate with each other via a packet based *communication subsystem*, that provides an unreliable broadcast primitive. Crucial for the operation of a CRA is, that transmission delays are bounded. The following two assumptions specify our communication subsystem in a formal way.

Assumption 4 (Broadcast Characteristics) *The nodes in our system communicate by message exchange using a broadcast operation. Two implementations are possible:*

- *In a* fully connected point-to-point network *by a sequence of send operations, whereby the maximum broadcast operation delay $\tau_{\max}$ characterizes the worst-case time to send out all messages.*
- *In a* broadcast-type network *by pertinent hardware, whereby the maximum broadcast latency $\lambda_{\max}$ characterizes the worst-case time between initiating and actually sending the message.*

Assumption 5 (Transmission Characteristics) *The nodes in our system communicate by message exchange featuring synchronous behavior. If no transmission faults occur, a message from node q to node p experiences a delay $\Delta t'_{p,q}$ subject to the delay condition*

$$\Delta t_{p,q} - \epsilon^{-}_{p,q} \leq \Delta t'_{p,q} \leq \Delta t_{p,q} + \epsilon^{+}_{p,q}, \tag{7}$$

where $\Delta t_{p,q}$ represents the deterministic part and $\epsilon^{\pm}_{p,q}$ the delivery uncertainties. For simplicity we will work with $\epsilon_{\max} = max_{p,q}\{\epsilon^{-}_{p,q} + \epsilon^{+}_{p,q}\}$ and $\Delta t_{\max} = max_{p,q}\{\Delta t_{p,q}\}$. For logical reasons we require $min_{p,q}\{\Delta t_{p,q}\} > \epsilon_{\max}$.

2.5 Faults

The above made system assumptions are only meaningful in the absence of faults. However, when dealing with a realistic system, faults may occur and need to be considered both for developing and analyzing distributed algorithms. Faults can affect the clocks (e.g. stuck, jump, rate error), the processors (e.g. various crashes) or the communication subsystem (e.g. omissions, timing errors, value errors). Therefore, it becomes necessary to set up a proper *fault model $\mathcal{F}$*, which specifies the prospective faults in our system. Section 4.4 will elaborate on these issues, aiming at an abstract treatment to keep our framework for clock rate synchronization as generic as possible.

3 Elements of Clock Rate Synchronization

This section presents the basic concepts for clock rate synchronization by introducing the notations and definitions along with technical lemmas. They will be the building blocks for the algorithm in Section 4 and its analysis in Section 5.

3.1 Interval Paradigm

As it turns out *asymmetric intervals* will play an important role in both describing and analyzing the algorithm, cf. [SS97] or earlier [Sch94], [Lam87] and [MO83]. We denote them with bold capital letters such as $\mathbf{I} = [x, r, y]$, where r is called the *reference point*, $x \geq 0$ the *left length*, $y \geq 0$ the *right length*. Such an asymmetric interval translates into a regular one $[r - x, r + y]$, where $(r - x)$ is the *left edge* and $(r + y)$ the *right edge*. In case of $x = y$ we also write $[r \pm x]$. An ordered set of them is written by calligraphic capital letters such as $\boldsymbol{\mathcal{I}}$. Operations on them are defined in a straightforward manner, summarized in the following Definition.

Definition 3 (Operations on Asymmetric Intervals) *Given two asymmetric intervals* $\mathbf{I}_1 = [x_1, r_1, y_1]$ *and* $\mathbf{I}_2 = [x_2, r_2, y_2]$. *We define the*

- reference point *as* $ref(\mathbf{I}_1) = r_1$,
- alignment *as* $align(\mathbf{I}_1) = \mathbf{I}_1 - ref(\mathbf{I}_1)$,
- right edge *as* $right(\mathbf{I}_1) = r_1 + y_1$,
- left edge *as* $left(\mathbf{I}_1) = r_1 - x_1$,
- length $\|\mathbf{I}_1\|$ *as* $x_1 + y_1$,
- exchange of lengths *as* $swap(\mathbf{I}_1) = [y_1, r_1, x_1]$.
- sum $\mathbf{I}_1 + \mathbf{I}_2$ *as* $[x_1 + x_2, r_1 + r_2, y_1 + y_2]$,
- scalar product $s\mathbf{I}_1$ *as* $[sx_1, sr_1, sy_1]$ *for some scalar* s,
- interval product $\mathbf{I}_1 \cdot \mathbf{I}_2$ *as* $[r_1x_2 + r_2x_1 - x_1x_2, r_1r_2, r_1y_2 + r_2y_1 + y_1y_2]$,
- intersection $\mathbf{I}_1 \cap \mathbf{I}_2$ *as* $[max\{r_1 - x_1, r_2 - x_2\}, min\{r_1 + y_1, r_2 + y_2\}]$,
- union $\mathbf{I}_1 \cup \mathbf{I}_2$ *as* $[min\{r_1 - x_1, r_2 - x_2\}, max\{r_1 + y_1, r_2 + y_2\}]$,

Note that in case of disjunct intervals the intersection delivers the *empty interval* $\emptyset$ and the union the closure of them. Furthermore, no reference point is explicitly given for these two operations, since several definitions are feasible.

3.2 Local Rate Intervals

For rate synchronization we have to find a way to capture the rate v_p of clock $\mathcal{C}_p$. Unfortunately, the rate of a clock cannot be observed directly, but we can postulate an asymmetric interval with reference point r_p and sufficiently long left/right lengths to include the ideal rate of 1. Both lengths and the reference point are given in multiples of the clock rate, hence $1 \in [v_p\theta_p^-, v_pr_p, v_p\theta_p^+]$, where θ_p^- and θ_p^+ are called *rate drifts*. Note that this introduces a relative expression of clocks rate as opposed to an absolute expression of clock states. Loosely speaking, "rates" are always regarded as the ideal rate 1 altered by the some "drifts". Dropping v_p leads to the definition of a *rate interval* $\mathbf{R}_p = [\theta_p^-, r_p, \theta_p^+]$, which contains enough information to run a clock rate algorithm.

Definition 4 (Properties of Rate Intervals) *A rate interval $\mathbf{R}_p$ is correct during a non-empty real-time period $\mathbf{T}$ iff $1 \in v_p(t)\mathbf{R}_p\ \forall\ t \in \mathbf{T}$. Two rate intervals $\mathbf{R}_p$ and $\mathbf{R}_q$ are consistent during a non-empty real-time period $\mathbf{T}$ iff $v_p(t)\mathbf{R}_p \bigcap v_q(t)\mathbf{R}_q \neq \emptyset\ \forall t \in \mathbf{T}$.*

In the special case that a rate interval $\mathbf{R}_p$ has 1 as reference point, we call it a *local rate interval.* These kind of rate intervals have great importance, since they can be maintained locally and possess many useful properties. The following Lemma asserts, that there exists a correspondence between local rate intervals and clock rates.

Lemma 1 (Local Rate Interval vs. Clock Drift) *If clock $\mathcal{C}_p$ has clock drift δ_p then $\mathbf{R}_p = [\delta_p/(1+\delta_p), 1, \delta_p/(1-\delta_p)]$ is the smallest correct local rate interval. On the contrary, if $\mathbf{R}_p = [\theta_p^-, 1, \theta_p^+]$ is a correct local rate interval for clock $\mathcal{C}_p$ then $\delta_p = max\{\theta_p^-/(1-\theta_p^-), \theta_p^+/(1+\theta_p^+)\}$ is the smallest clock drift.*

Proof For the first part of the lemma, we have to determine the smallest rate drifts θ_p^- and θ_p^+ that satisfy

$$v_p(1-\theta_p^-) \leq 1 \leq v_p(1+\theta_p^+) \tag{8}$$

from Definition 4. Definition 2 assures that clock rate $v_p \in [1-\delta_p, 1+\delta_p]$. To find the smallest θ_p^- and θ_p^+, it is sufficient to draw upon the extreme values of v_p. Hence, plugging in $v_p = 1+\delta_p$ in (8) yields $\theta_p^- \geq \delta_p/(1+\delta_p)$ and $\theta_p^+ \geq 0$, and $v_p = 1-\delta_p$ delivers $\theta_p^- \geq 0$ and $\theta_p^+ \geq \delta_p/(1-\delta_p)$. Putting these statements together, we obtain the desired local rate interval

$$\mathbf{R}_p = \left[max\{0, \frac{\delta_p}{1+\delta_p}\}, 1, max\{0, \frac{\delta_p}{1-\delta_p}\}\right].$$

For the second part of the lemma, we have to find the smallest δ_p that matches the possible clock rates v_p induced by the local rate interval $\mathbf{R}_p$. For that purpose we transform (8) into

$$-\frac{\theta_p^+}{1+\theta_p^+} \leq v_p - 1 \leq \frac{\theta_p^-}{1-\theta_p^-}, \tag{9}$$

which can be rewritten as $|v_p - 1| \leq max\{\frac{\theta_p^-}{1-\theta_p^-}, \frac{\theta_p^+}{1+\theta_p^+}\} = \delta_p$. □

It is interesting to note, that local rate intervals exhibit an intrinsic asymmetry. The first part of Lemma 1 together with a specified oscillator drift ρ_p can be used to initialize local rate intervals, thus

$$\mathbf{R}_p(t_0) = \left[\frac{\rho_p}{1+\rho_p}, 1, \frac{\rho_p}{1-\rho_p}\right] \tag{10}$$

by a neutral setting of the coupling factor and if t_0 denotes the real-time of initialization.

For further considerations it is important to relate an observable duration ΔT on a local clock with their real-time counterpart Δt and vice versa. Our next Lemma establishes these relationships.

Lemma 2 (Duration Estimation) *Given a clock $\mathcal{C}_p$ paced by an oscillator with stability σ_p. Let t_1 resp. t_2 be real times and $T_1 = C_p(t_1)$ resp. $T_2 = C_p(t_2)$ the corresponding clock states, where $t_1 \leq t_2$ and no resynchronization occurred in between. If clock $\mathcal{C}_p$ has rate $v_p(t_1)$ at t_1 then we have*

$$v_p(t_1)\left((t_2-t_1) - \frac{\sigma_p}{2}(t_2-t_1)^2\right) \leq T_2 - T_1 \leq v_p(t_1)\left((t_2-t_1) + \frac{\sigma_p}{2}(t_2-t_1)^2\right)$$

and the converse

$$\frac{T_2-T_1}{v_p(t_1)}-\frac{\sigma_p(T_2-T_1)^2}{2v_p^2(t_1)}-\mathcal{O}\left(\sigma_p^2(T_2-T_1)^3\right)\le t_2-t_1\le\frac{T_2-T_1}{v_p(t_1)}+\frac{\sigma_p(T_2-T_1)^2}{2v_p^2(t_1)}+\mathcal{O}\left(\sigma_p^2(T_2-T_1)^3\right).$$

Proof We introduced the clock rate $v_p(t)$ as the derivative of the time dependable function $C_p(t)$ of the clock state. Applying the integral from starting point t_1 to successive point $t_1+\Delta t$, we get

$$\Delta T = C_p(t_1+\Delta t)-C_p(t_1)=\int_0^{\Delta t} v_p(t_1+\xi)d\xi. \tag{11}$$

In the absence of resynchronizations we derive from the stability condition of Assumption 1 that the clock rate at $t_1+\xi$ satisfies

$$v_p(t_1)(1-\sigma_p\xi)\le v_p(t_1+\xi)\le v_p(t_1)(1+\sigma_p\xi) \tag{12}$$

for any $\xi\ge 0$. Using these relations as majorants for the integrand in (11) and relying on the non-accumulating nature of short-term violations, we can bound the clock state difference by

$$\begin{aligned}\int_0^{\Delta t} v_p(t_1)(1-\sigma_p\xi)d\xi \quad &\le \Delta T \le \quad \int_0^{\Delta t} v_p(t_1)(1+\sigma_p\xi)d\xi\\ v_p(t_1)\left(\Delta t-\frac{\sigma_p}{2}\Delta t^2\right) \quad &\le \Delta T \le \quad v_p(t_1)\left(\Delta t+\frac{\sigma_p}{2}\Delta t^2\right),\end{aligned}$$

which proves the first part of the Lemma setting $\Delta t = t_2-t_1$.

For the second part we treat the above relation as a quadratic equation and choose the corresponding roots in order to find bounds on Δt. This leads to

$$\frac{1}{\sigma_p}\left[-1+\sqrt{1+\frac{2\sigma_p\Delta T}{v_p(t_1)}}\right]\le\Delta t\le\frac{1}{\sigma_p}\left[1-\sqrt{1-\frac{2\sigma_p\Delta T}{v_p(t_1)}}\right],$$

where in our setting the second term under each root is small compared to 1. To simplify the bounds we use the asymptotic approximation valid for $x\to 0$

$$\sqrt{1\pm x}=1\pm\frac{x}{2}\mp\frac{x^2}{8}+\mathcal{O}\left(x^3\right),$$

and obtain after some algebraic manipulations that

$$\frac{\Delta T}{v_p(t_1)}-\frac{\sigma_p\Delta T^2}{2v_p^2(t_1)}+\mathcal{O}\left(\frac{\sigma_p^2\Delta T^3}{v_p^3(t_1)}\right)\le\Delta t\le\frac{\Delta T}{v_p(t_1)}+\frac{\sigma_p\Delta T^2}{2v_p^2(t_1)}+\mathcal{O}\left(\frac{\sigma_p^2\Delta T^3}{v_p^3(t_1)}\right).$$

Note that the linear terms represent the bounds in case of stable oscillators. Since clock rates v_p are very close to 1, we drop them and get the simplified $\mathcal{O}$-terms of the Lemma. □

Each node p maintains a local rate interval $\mathbf{R}_p$, which should remain correct as time proceeds. However, the oscillator stability requires to deteriorate them as stated in the following Lemma.

Lemma 3 (Deterioration of Rate Intervals) *Given a clock $\mathcal{C}_p$ paced by an oscillator with stability σ_p and maximum drift ρ_p. If $\mathbf{R}_p$ denotes a correct rate interval at T_1, then*

$$\mathbf{R}_p + \left[0 \pm ref(\mathbf{R}_p)\frac{\sigma_p(T_2 - T_1)}{1 - 2\rho_p}\right] + \left[0 \pm \mathcal{O}\left(\sigma_p||\mathbf{R}_p||(T_2 - T_1) + \sigma_p^2(T_2 - T_1)^2\right)\right]$$

is correct at $T_2 \geq T_1$, when no resynchronizations occurred in between.

Proof Due to the correctness of $\mathbf{R}_p = [\theta_p^-, r_p, \theta_p^+]$ at $T_1 = C_p(t_1)$ we know from Definition 4 that

$$v_p(t_1)(r_p - \theta_p^-) \leq 1 \leq v_p(t_1)(r_p + \theta_p^+). \tag{13}$$

In the absence of resynchronizations until $T_2 = C_p(t_2)$, the stability properties according to Assumption 1 ensures that $|\frac{v_p(t_2)}{v_p(t_1)} - 1| \leq \sigma_p(t_2 - t_1)$. Making $v_p(t_1)$ explicit and using the asymptotic approximation $(1 \pm x)^{-1} = 1 \mp x + \mathcal{O}(x^2)$ valid for $x \to 0$ yields

$$v_p(t_2)\left(1 - \sigma_p(t_2 - t_1) - \mathcal{O}\left(\sigma_p^2(t_2 - t_1)^2\right)\right) \leq v_p(t_1) \leq v_p(t_2)\left(1 + \sigma_p(t_2 - t_1) + \mathcal{O}\left(\sigma_p^2(t_2 - t_1)^2\right)\right).$$

To bring in the observable duration $T_2 - T_1$, we consult Lemma 2 to get an upper bound on $t_2 - t_1$, hence

$$v_p(t_2)\left(1 - \frac{\sigma_p(T_2 - T_1)}{v_p(t_1)} - \mathcal{O}\left(\sigma_p^2(T_2 - T_1)^2\right)\right) \leq v_p(t_1) \tag{14}$$

and

$$v_p(t_2)\left(1 + \frac{\sigma_p(T_2 - T_1)}{v_p(t_1)} + \mathcal{O}\left(\sigma_p^2(T_2 - t_1)^2\right)\right) \geq v_p(t_1). \tag{15}$$

Before plugging $v_p(t_1)$ from (14) and (15) into (13), we replace the denominator in the error term by $(1 - 2\rho_p)$ according to (2). This leads eventually to

$$v_p(t_2)\left(r_p - \theta_p^- - r_p\frac{\sigma_p(T_2 - T_1)}{1 - 2\rho_p} - \mathcal{O}\left(\sigma_p\theta_p^-(T_2 - T_1) + \sigma_p^2(T_2 - T_1)^2\right)\right) \leq 1$$

and

$$v_p(t_2)\left(r_p + \theta_p^+ + r_p\frac{\sigma_p(T_2 - T_1)}{1 - 2\rho_p} + \mathcal{O}\left(\sigma_p\theta_p^+(T_2 - T_1) + \sigma_p^2(T_2 - T_1)^2\right)\right) \geq 1$$

which proofs the deterioration of rate intervals. □

As a trivial consequence, local rate intervals $\mathbf{R}_p$ can be deteriorated by $[0 \pm \sigma_p(T_2 - T_1)/(1 - 2\rho_p)]$ when neglecting the $\mathcal{O}$-terms.

3.3 Consonance Intervals

For internal rate synchronization we have express consonance, which measures how close the clock rates are together. Returning to the definition of rate intervals, we introduce consonance by imposing distances upon their reference points.

Definition 5 ($\boldsymbol{\gamma}$-Consonance) *Given consonance interval $\boldsymbol{\gamma} = [\gamma^-, 0, \gamma^+]$ with $0 \leq \gamma^-, \gamma^+ \ll 1$ and an ensemble of clocks $\mathcal{C}_1, \ldots, \mathcal{C}_n$ with their rates $v_1(t), \ldots, v_n(t)$. An associated set of correct rate intervals $\boldsymbol{\mathcal{R}} = \{\mathbf{R}_1, \ldots, \mathbf{R}_n\}$ is called $\boldsymbol{\gamma}$-consonant iff $\bigcap_{i=1}^n v_i(t)(ref(\mathbf{R}_i) + \boldsymbol{\gamma}) \neq \emptyset$.*

As before, local rate intervals play a special role for establishing a relationship between the $\boldsymbol{\gamma}$-consonance property and the consonance γ of the associated ensemble.

Lemma 4 ($\boldsymbol{\gamma}$-Consonance vs. Consonance γ) *Given an ensemble of clocks $\mathcal{C}_1, \ldots, \mathcal{C}_n$ with a maximum clock drift δ. If such an ensemble has consonance γ, then a set $\boldsymbol{\mathcal{R}} = \{\mathbf{R}_1, \ldots, \mathbf{R}_n\}$ of local rate intervals is $\boldsymbol{\gamma}$-consonant for any $\boldsymbol{\gamma} \supseteq [0 \pm \gamma/(2(1-\delta))]$. On the contrary, if a set $\boldsymbol{\mathcal{R}} = \{\mathbf{R}_1, \ldots, \mathbf{R}_n\}$ of local rate intervals is $\boldsymbol{\gamma}$-consonant, then they have consonance $\gamma = (1+\delta)||\boldsymbol{\gamma}||$.*

Proof For the first part of the Lemma we have to ensure that $v_p(1+\boldsymbol{\gamma})$ and $v_q(1+\boldsymbol{\gamma})$ intersect, since $ref(\mathbf{R}_p) = ref(\mathbf{R}_q) = 1$. Therefore making $\boldsymbol{\gamma} = [\gamma^-, 0, \gamma^+]$ sufficiently long, in particular, $v_p\gamma^+ + v_q\gamma^- \geq \gamma$ has to hold if $v_p \leq v_q$, and $v_p\gamma^- + v_q\gamma^+ \geq \gamma$ if $v_p \geq v_q$. By solving these unequations we easily get

$$\gamma^-, \gamma^+ \geq \frac{\gamma}{v_p + v_q},$$

and noting that the sum of any two clock rates cannot fall short of $2(1-\delta)$ we are done.

For the second part of the Lemma we know from the non-empty intersection of $v_p(1+\boldsymbol{\gamma})$ and $v_q(1+\boldsymbol{\gamma})$ that $v_p - v_q \leq v_p\gamma^- + v_q\gamma^+$ if $v_p \geq v_q$, or $v_q - v_p \leq v_p\gamma^+ + v_q\gamma^-$ if $v_p \geq v_q$. Combining them and recalling that no clock rate exceeds $1+\delta$ leads to

$$\begin{aligned} |v_p - v_q| &\leq max\{v_p\gamma^- + v_q\gamma^+, v_p\gamma^+ + v_q\gamma^-\} \\ &= \gamma^- max\{v_p, v_q\} + \gamma^+ max\{v_p, v_q\} \\ &\leq (1+\delta)||\boldsymbol{\gamma}||. \end{aligned}$$

□

Corollary 1 ($\boldsymbol{\gamma}$-Consonance by Clock Drift) *If an ensemble of clocks $\mathcal{C}_1, \ldots, \mathcal{C}_n$ has a maximum clock drift δ, then a set $\boldsymbol{\mathcal{R}} = \{\mathbf{R}_1, \ldots, \mathbf{R}_n\}$ of correct local rate intervals is $\boldsymbol{\gamma}$-consonant for any $\boldsymbol{\gamma} \supseteq [0 \pm \delta/(1-\delta)]$.*

Proof Immediately from the first part of Lemma 4 by setting $\gamma = 2\delta$. □

So far we are able to express the quality of internal rate synchronization, but in order to get a handle on the operation of an internal CRA, we need to introduce the notion of *internal global rate* $\omega(t)$. The idea is to define $\omega(t)$ in such a way that local rate intervals satisfy $\omega(t) \in \bigcap_{i=1}^n v_i(t)(1+\boldsymbol{\gamma})\ \forall t \geq t_0$, which guarantees a particular consonance according to Lemma 4. Section 5.1 will show that $\omega(t)$ is a piecewise constant function in the proximity of the ideal rate 1. This leads eventually to the definition of $\boldsymbol{\gamma}$-correctness.

Definition 6 ($\boldsymbol{\gamma}$-Correctness) *Given consonance interval $\boldsymbol{\gamma} = [\gamma^-, 0, \gamma^+]$ with $0 \leq \gamma_p^-, \gamma_p^+ \ll 1$ and a clock $\mathcal{C}_p$ with its rate $v_p(t)$. A rate interval $\mathbf{R}_p$ is called $\boldsymbol{\gamma}_p$-correct w.r.t. internal global rate $\omega(t)$ at t iff $\omega(t) \in v_p(t)(ref(\mathbf{R}) + \boldsymbol{\gamma}_p)$. For an ensemble of clocks $\mathcal{C}_1, \ldots, \mathcal{C}_n$ with their rates $v_1(t), \ldots, v_n(t)$ the associated set of correct rate intervals $\boldsymbol{\mathcal{R}} = \{\mathbf{R}_1, \ldots, \mathbf{R}_n\}$ is called $\boldsymbol{\gamma}$-correct w.r.t. internal global rate $\omega(t)$ at t iff each of them is $\boldsymbol{\gamma}$-correct, thus $\omega(t) \in v_i(t)(ref(\mathbf{R}_i) + \boldsymbol{\gamma})$ for all $1 \leq i \leq n$.*

Lemma 5 ($\boldsymbol{\gamma}$-Correctness of an ensemble) *If clocks $\mathcal{C}_p$ are $\boldsymbol{\gamma}_p$-correct w.r.t. internal global rate $\omega(t)$, respectively, then the ensemble is $\boldsymbol{\gamma}$-correct for any $\boldsymbol{\gamma} \supseteq \cup_{i=1}^n \boldsymbol{\gamma}_i$.*

Proof Let $\boldsymbol{\gamma}$ be an interval such that $\boldsymbol{\gamma} \supseteq \cup_{i=1}^n \boldsymbol{\gamma}_i$. Suppose $\bigcap_{i=1}^n v_i \boldsymbol{\gamma} = \emptyset$, then there exist at least two disjunct intervals $v_p\boldsymbol{\gamma}$ and $v_q\boldsymbol{\gamma}$. Since $\boldsymbol{\gamma}_p, \boldsymbol{\gamma}_q \subseteq \boldsymbol{\gamma}$ we conclude that $v_p\boldsymbol{\gamma}_p \cap v_q\boldsymbol{\gamma}_q = \emptyset$, which contradicts with the correctness assumptions of the clocks. □

By putting Lemma 5 and 4 back to back, we know that if each clock is $\boldsymbol{\gamma}_p$-correct, then the ensemble has consonance $\gamma = 2(1+\delta)max_{i=1}^{n}\ ||\boldsymbol{\gamma}_i||$, since $\boldsymbol{\gamma}$-correctness implies $\boldsymbol{\gamma}$-consonance but not necessarily the converse. This result is the key to the analysis of the internal CRA. Note further, that the introduction of internal global rate is purely artificial, but it allows us to reason about consonance by considering each clock separately, which greatly simplifies the analysis.

Lemma 6 (Deterioration of Consonance Intervals) *Given a clock $\mathcal{C}_p$ paced by an oscillator with stability σ_p and maximum drift ρ_p. If $\mathbf{R}_p$ is $\boldsymbol{\gamma}_p$-correct at T_1, then it is $\boldsymbol{\gamma}'_p$-correct at $T_2 \geq T_1$, where*

$$\boldsymbol{\gamma}'_p = \boldsymbol{\gamma}_p + \left[0 \pm ref(\mathbf{R}_p)\frac{\sigma_p(T_2-T_1)}{1-2\rho_p}\right] + \left[0 \pm \mathcal{O}\left(\sigma_p||\boldsymbol{\gamma}_p||(T_2-T_1)+\sigma_p^2(T_2-T_1)^2\right)\right]$$

and no resynchronizations occurred in between.

Proof The same line of reasoning as in the proof for Lemma 3 can be applied, but instead of the ideal rate 1 the internal global rate $\omega(t)$ is enclosed, which is identical at T_1 and T_2 if no resynchronizations take place. □

Figure 2 illustrates the deterioration of rate and consonance intervals. Starting out at t_1 the local rate interval $\mathbf{R}_p$ satisfies $1 \in v_p(t_1)\mathbf{R}_p(t_1)$, and the consonance interval $\boldsymbol{\gamma}$ satisfies $\omega^k \in v_p(t_1)(ref(\mathbf{R}_p(t_1))+\boldsymbol{\gamma})$, respectively. The cone formed by the dotted lines represent the feasible clock rates according to the stability condition of the underlying oscillator, whereas the curved line represents an exemplary progress of the clock rate. At t_2 the intervals are enlarged such that they remain correct.

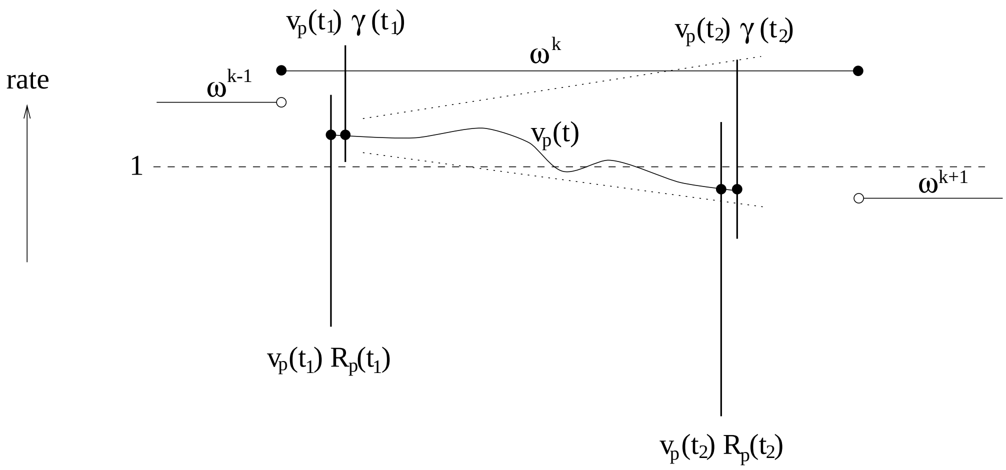

Figure 2: Deterioration of Rate and Consonance Intervals

3.4 Rate Measurement

Any clock rate algorithm has to work on the grounds of relative rates between clock pairs. This restriction comes from the fact that a local clock is not able to determine it's rate by itself, otherwise rate synchronization would be trivial. We capture the relative rate of a remote clock against a local one with the help of a *quotient rate interval* $\mathbf{Q}_{p,q}$. The indexing follows the rule that the first one denotes the local node and the second the remote one. Such an interval quantifies how fast or slow a remote clock is in respect to it's own.

Definition 7 (Quotient Rate Interval) *Given two clocks $\mathcal{C}_p$ resp. $\mathcal{C}_q$ with their rates $v_p(t')$ resp. $v_q(t'')$ during the non-empty real-time periods $\mathbf{T}_p$ resp. $\mathbf{T}_q$. An asymmetric interval $\mathbf{Q}_{p,q}$ that holds $\frac{v_q(t')}{v_p(t'')} \in \mathbf{Q}_{p,q}\ \forall t' \in \mathbf{T}_q\ \forall t'' \in \mathbf{T}_p$ is called a quotient rate interval of remote clock $\mathcal{C}_q$ during $\mathbf{T}_q$ against local clock $\mathcal{C}_p$ during $\mathbf{T}_p$.*

To obtain quotient rate intervals $\mathbf{Q}_{p,q}$ we carry out a simple protocol as illustrated in Figure 3. It is based on repeated message pairs, such that remote node q broadcasts periodically a message that contains the latest clock state of $\mathcal{C}_q$. The receiving node p records the current clock state of $\mathcal{C}_p$ upon message arrival. From Figure 3 we extract clock states T_q, T_p from the first message M_p, and T'_q, T'_p from the second message M'_p. With these four clock states and taking into account the transmission characteristics from Assumption 5, node p can compute quotient rate intervals as certified by Lemma 8. For preparation, we need the following technical Lemma.

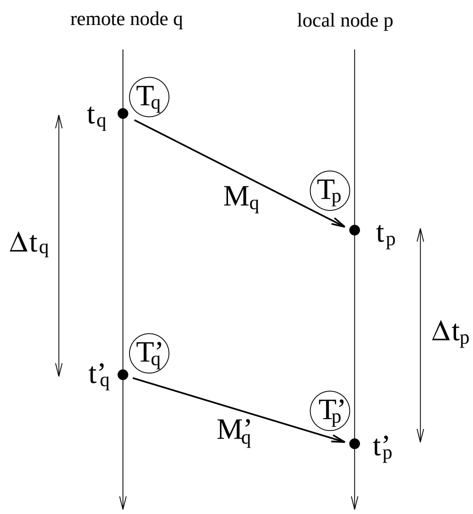

Figure 3: Protocol for Rate Measuring

Lemma 7 (Min/Max Clock Rate) *Given a clock $\mathcal{C}_p$ paced by an oscillator with stability σ_p and let t_1 resp. t_2 be real times and $T_1 = C_p(t_1)$ resp. $T_2 = C_p(t_2)$ the corresponding clock states, where $t_1 \le t_2$. When no resynchronization occurred in between, then clock rate $v_p(t)$ obeys*

$$v_p(t) \in \frac{T_2 - T_1}{t_2 - t_1}\left[1 \pm \frac{\sigma_p}{2}(t_2 - t_1)\right] + \left[0 \pm \mathcal{O}\left(\sigma_p^2 (t_2 - t_1)^2\right)\right]$$

for any $t \in [t_1, t_2]$.

Proof We have to consider two cases, either where the clock rate increases or decreases maximally during the real-time duration $\Delta t = t_2 - t_1$ and the expired logical time still amounts to $\Delta T = T_2 - T_1$. The actual clock rate lies somewhere between these two extremes, which gives raise for bounding $v_p(t)\ \forall t \in [t_1, t_2]$. In the first case (increasing rate), the clock has a minimum rate $v'_{\min}$ at t_1 related by

$$\Delta T = v'_{\min}\left(\Delta t + \frac{\sigma_p}{2}(\Delta t)^2\right) \tag{16}$$

according to Lemma 2. By virtue of Assumption 1 the maximum rate $v'_{\max}$ at t_2 is given by

$$v'_{\max} = v'_{\min}(1 + \sigma_p \Delta t). \tag{17}$$

For the second case (decreasing rate) we find similar expressions for the maximum rate

$$\Delta T = v''_{\max}\left(\Delta t - \frac{\sigma_p}{2}(\Delta t)^2\right) \tag{18}$$

and for the minimum rate

$$v''_{\min} = v''_{\max}(1 - \sigma_p \Delta t). \tag{19}$$

We can easily show that $v'_{\max} \leq v''_{\max}$ and $v'_{\min} \geq v''_{\min}$, but the asymptotic approximation $(1 \pm x)^{-1} = 1 \mp x + \mathcal{O}(x^2)$ valid for $x \to 0$ vanishes the differences. Hence, using (16) for the minimum rate bound yields

$$v_p(t) \geq \frac{\Delta T}{\Delta t} \cdot \left(1 - \frac{\sigma_p}{2}\Delta t\right) + \mathcal{O}\left(\sigma_p^2 \Delta t^2\right) \cdot \frac{\Delta T}{\Delta t}$$

and (18) for the maximum rate bound

$$v_p(t) \leq \frac{\Delta T}{\Delta t} \cdot \left(1 + \frac{\sigma_p}{2}\Delta t\right) + \mathcal{O}\left(\sigma_p^2 \Delta t^2\right) \cdot \frac{\Delta T}{\Delta t}$$

both for $t \in [t_1, t_2]$, which completes the proof. □

Lemma 8 (Rate Measurement) *Let $\mathcal{C}_p$ and $\mathcal{C}_q$ be clocks driven by oscillators with stability σ_p resp. σ_q, and maximum drift ρ_p resp. ρ_q. By executing the above sketched protocol in Figure 3 without any resynchronizations in between, interval*

$$\mathbf{Q}_{p,q} = \left[\frac{T'_q - T_q}{T'_p - T_p} \pm \left(\frac{(\sigma_p + \sigma_q)(T'_q - T_q)}{2(1 - 2\rho_q)} + \frac{\epsilon_{\max}(1 + 2\rho_p)}{T'_p - T_p}\right)\right] + \left[0 \pm \mathcal{O}\left((\sigma_p^2 + \sigma_q^2)\Delta T_p^2 + \sigma_q \epsilon_{\max}\right)\right]$$

is a quotient rate interval during the real-time periods $[t_p, t'_p]$ resp. $[t_q, t'_q]$.

Proof First of all, we want to find a relation between the involved real-time periods $\Delta t_q = t'_q - t_q$ at the sending side, and $\Delta t_p = t'_p - t_p$ at the receiving side. Due to Assumption 5 the delivery delay for the first message M_p can be captured by $\Delta t_{p,q} - \epsilon^-_{p,q} \leq t_p - t_q \leq \Delta t_{p,q} + \epsilon^+_{p,q}$, and for the second message M'_p by $\Delta t_{p,q} - \epsilon^-_{p,q} \leq t'_p - t'_q \leq \Delta t_{p,q} + \epsilon^+_{p,q}$. Subtracting them and further algebraic manipulations deliver

$$|\Delta t_q - \Delta t_p| \leq \epsilon^-_{p,q} + \epsilon^+_{p,q} \leq \epsilon_{\max}. \tag{20}$$

Next we want to derive a lower bound on the ratio $\frac{v_q(t')}{v_p(t'')}$, where $t' \in [t_q, t'_q]$ and $t'' \in [t_p, t'_p]$. From Lemma 7 we know that the maximum clock rate $v_p^{\max}$ at node p during $[t_p, t'_p]$ is related by

$$\Delta t_p = \frac{\Delta T_p}{v_p^{\max} - \frac{\sigma_p}{2}\Delta T_p - \mathcal{O}\left(\sigma_p^2 \Delta t_p^2\right)} \tag{21}$$

and the minimum clock rate $v_q^{\min}$ at node q during $[t_q, t'_q]$ by

$$\Delta t_q = \frac{\Delta T_q}{v_q^{\min} + \frac{\sigma_q}{2}\Delta T_q + \mathcal{O}\left(\sigma_q^2 \Delta t_q^2\right)}. \tag{22}$$

Putting together (21) and (22) via (20) yields

$$\begin{aligned}\Delta T_p v_q^{\min} - \Delta T_q v_p^{\max} \quad &\geq \quad -\frac{\sigma_p + \sigma_q}{2}\Delta T_p \Delta T_q - \mathcal{O}\left(\sigma_q^2 \Delta t_q^2\right)\Delta T_p - \mathcal{O}\left(\sigma_p^2 \Delta t_p^2\right)\Delta T_q - \\ &\epsilon_{\max}\left(v_p^{\max} - \frac{\sigma_p}{2}\Delta T_p - \mathcal{O}\left(\sigma_p^2 \Delta t_p^2\right)\right)\left(v_q^{\min} + \frac{\sigma_q}{2}\Delta T_q + \mathcal{O}\left(\sigma_q^2 \Delta t_q^2\right)\right)\end{aligned}$$

and after some algebraic manipulations we get

$$\frac{v_q^{\min}}{v_p^{\max}} \geq \frac{\Delta T_q}{\Delta T_p}\left(1 - \frac{(\sigma_p + \sigma_q)\Delta T_p}{2 v_p^{\max}}\right) - \frac{\epsilon_{\max} v_q^{\min}}{\Delta T_q} - \mathcal{O}\left(\sigma_p^2 \Delta t_p^2 + \sigma_q^2 \Delta t_q^2 + \sigma_q \epsilon_{\max}\right). \tag{23}$$

To get rid of $v_p^{\max}$ resp. $v_q^{\min}$ in the error terms, we make use of the maximum oscillator drift ρ_p resp. ρ_q of a non-faulty clock and the agreed condition (2). Inside the $\mathcal{O}$-term we can replace Δt_p^2 and Δt_q^2 by ΔT_p^2 according to Lemma 2. Thus, we can transform (23) into

$$\frac{v_q(t')}{v_p(t'')} \geq \frac{\Delta T_q}{\Delta T_p}\left(1 - \frac{(\sigma_p + \sigma_q)\Delta T_p}{2(1 - 2\rho_p)} - \frac{\epsilon_{\max}(1 + 2\rho_q)}{\Delta T_p}\right) - \mathcal{O}\left((\sigma_p^2 + \sigma_q^2)\Delta T_p^2 + \sigma_q \epsilon_{\max}\right)$$

valid for all $t' \in [t_q, t_q']$ and $t'' \in [t_p, t_p']$. A similar line of reasoning starting out with

$$\Delta t_p = \frac{\Delta T_p}{v_p^{\min} + \frac{\sigma_p}{2}\Delta T_p + \mathcal{O}\left(\sigma_p^2 \Delta t_p^2\right)}$$

and

$$\Delta t_q = \frac{\Delta T_q}{v_q^{\max} - \frac{\sigma_q}{2}\Delta T_q - \mathcal{O}\left(\sigma_q^2 \Delta t_q^2\right)}$$

brings out the desired upper bound on $\frac{v_q(t')}{v_p(t'')}$, which completes the proof. □

Remarks

- Since rate measurements will take place less frequently than state resynchronizations, we have to care about interspersed state corrections to get useful clock state differences ΔT_p and ΔT_q. In fact, we have to maintain and subsequently exchange the sum of state corrections on both sides, irrespective of being applied instantaneously or by continuous amortization.

- The ratio $\frac{\Delta T_q}{\Delta T_p}$ is fairly good indicator if clocks are running properly. Values too far away from 1 allows us to take away faulty once by an easy sanity check, otherwise fault assumptions have to capture them.

- Rate measurement is the means where the information of the clock rates come in, but it requires the cooperation of both nodes to compute the quotient as opposed from a CSA. Straight from Lemma 8 and dropping delivery uncertainties we get

$$\mathbf{Q}_{p,p} = \left[1 \pm \frac{\sigma_p \Delta T_p}{1 - 2\rho_p}\right] + \left[0 \pm \mathcal{O}\left(\sigma_p^2 \Delta T_p^2\right)\right] \tag{24}$$

for a common period ΔT_p, reflecting again the impossibility to acquire information about the own clock rate.

- By taking a look at the formula for the quotient rate interval in Lemma 8, we observe that for stable clocks the interval degrades to a point if the measurement duration becomes large. This is not the case for unstable clocks, so we can imagine a certain rate resynchronization period, given roughly by

$$\sqrt{\frac{\epsilon_{\max}}{\sigma_{\max}}(1+4\rho_{\max})},$$

  where the interval length becomes smallest. For short rate resynchronization periods, the delivery uncertainty spoils the rate measurement. Note that this is different to an CSA, where apart from bandwidth concerns there is no particular lower limit on the state resynchronization period.

- Note that the deterministic part $\Delta t_{p,q}$ of the message delivery delay does not pop up in the formula, since it is included in the measured durations ΔT_p and ΔT_q.

3.5 Remote Rate Intervals

Suppose node q maintains a correct local rate interval $\mathbf{R}_q$ of its clock $\mathcal{C}_q$. For synchronization purpose, we want to transfer it correctly to another node p resulting in the *remote rate interval* $\mathbf{R}_{p,q}$. This operation is similar to a CSA, where an accuracy interval of one node is passed on to the others within the ensemble.

However, the way to carry over a rate interval from node q to a node p is more intricate than for a state interval, since switching correctness involves the relative rate measurement via quotient rate interval $\mathbf{Q}_{p,q}$. The importance of these intervals come from the fact, that they contain the ideal rate 1 expressed in the "rate world" of the particular local node. The following lemma pins down this property in a formal way.

Lemma 9 (Computing Remote Rate Intervals) *Let $\mathcal{C}_p$ and $\mathcal{C}_q$ be clocks driven by oscillators with stability σ_p and σ_q, respectively. The messages for the rate measurement are sent from node q at $T_q = C_q(t_q)$ resp. $T'_q = C_q(t'_q)$ and received at node p at $T_p = C_p(t_p)$ resp. $T'_p = C_p(t'_p)$ according to the transmission characteristics of Assumption 5, cf. Figure 3. If $\mathbf{R}_q$ is a correct local rate interval at t_q on remote node q, then*

$$\mathbf{R}_{p,q} = \mathbf{Q}_{p,q} \cdot \mathbf{R}_q$$

is a correct remote rate interval at t'_p on local node p, where $\mathbf{Q}_{p,q}$ is a quotient rate interval during the real-time periods $[t_p, t'_p]$ resp. $[t_q, t'_q]$.

Proof We string together the properties of $\mathbf{Q}_{p,q}$ and $\mathbf{R}_q$ forming the remote rate interval $\mathbf{R}_{p,q}$, and show that this leads to the desired interval multiplication.

From Definition 4 we know that the local rate interval $\mathbf{R}_q = [\theta_q^-, 1, \theta_q^+]$ of a remote clock $\mathcal{C}_q$ satisfies

$$v_q(t_q)(1-\theta_q^-) \le 1 \le v_q(t_q)(1+\theta_q^+), \tag{25}$$

and by specialization of Lemma 8 that the quotient rate interval $\mathbf{Q}_{p,q} = Q_{p,q}[u, 1, u]$, with $Q_{p,q} = \frac{T'_q - T_q}{T'_p - T_p}$ and $u = \frac{(\sigma_p+\sigma_q)(T'_p - T_p)}{2(1-2\rho_q)} + \frac{\epsilon_{\max}(1-2\rho_p)}{T'_q - T_q} + \mathcal{O}\Big((\sigma_p^2 + \sigma_q^2)\Delta T_p^2 + \sigma_q \epsilon_{\max}\Big)$, holds

$$Q_{p,q}(1-u) \le \frac{v_q(t_q)}{v_p(t'_p)} \le Q_{p,q}(1+u). \tag{26}$$

Plugging in $v_q(t_q)$ from (26) into (25) yields

$$v_p(t'_p)Q_{p,q}(1-u)(1-\theta_q^-) \leq 1 \leq v_p(t'_p)Q_{p,q}(1+u)(1+\theta_q^+).$$

We can translate the last equation into the asymmetric interval notation

$$1 \in v_p(t'_p)Q_{p,q}[u+\theta_q^- - u\theta_q^-, 1, u+\theta_q^+ - u\theta_q^+],$$

which is according to Definition 3 equivalent to

$$1 \in v_p(t'_p)Q_{p,q}[u,1,u] \cdot [\theta_q^-, 1, \theta_q^+].$$

Finally, a resubstitution provides the desired property

$$1 \in v_p(t'_p)(\mathbf{Q}_{p,q} \cdot \mathbf{R}_q).$$

□

In general, a remote rate interval doesn't have 1 as reference point, but $ref(\mathbf{R}_{p,q}) = ref(\mathbf{Q}_{p,q})$. A remote rate interval of itself becomes to $\mathbf{R}_{p,p} = \mathbf{Q}_{p,p} \cdot \mathbf{R}_p$ and using (24) proves

$$\begin{aligned}
\mathbf{R}_{p,p} &= \left[1 \pm \left(\frac{\sigma_p \Delta T_p}{1-2\rho_p} + \mathcal{O}\left(\sigma_p^2 \Delta T_p^2\right)\right)\right] \cdot [\theta_p^-, 1, \theta_p^+] \\
&= \left[\theta_p^- + \frac{(1-\theta_p^-)\sigma_p \Delta T_p}{1-2\rho_p} + \mathcal{O}\left(\sigma_p^2 \Delta T_p^2\right), 1, \theta_p^+ + \frac{(1+\theta_p^+)\sigma_p \Delta T_p}{1-2\rho_p} + \mathcal{O}\left(\sigma_p^2 \Delta T_p^2\right)\right] \\
&= \mathbf{R}_p + \left[0 \pm \frac{\sigma_p \Delta T_p}{1-2\rho_p}\right] + \left[0 \pm \mathcal{O}\left(\sigma_p ||\mathbf{R}_p|| \Delta T_p + \sigma_p^2 \Delta T_p^2\right)\right]. \qquad (27)
\end{aligned}$$

Note that this expression equals the deterioration formula for rate intervals as given in Lemma 3, which makes perfect sense because measuring the rate by itself over a particular duration means to cope with the oscillator stability.

Corollary 2 (Consonance of Remote Rate Intervals) *Let $\mathcal{C}_p$ and $\mathcal{C}_q$ be clocks driven by oscillators with stability σ_p and σ_q, respectively. The messages for the rate measurement are sent from node q at $T_q = C_q(t_q)$ resp. $T'_q = C_q(t'_q)$ and received at node p at $T_p = C_p(t_p)$ resp. $T'_p = C_p(t'_p)$ according to the transmission characteristics of Assumption 5, cf. Figure 3. Furthermore, let $\mathbf{Q}_{p,q}$ be a quotient rate interval during the real-time periods $[t_p, t'_p]$ resp. $[t_q, t'_q]$. If local rate interval $\mathbf{R}_q$ is $\boldsymbol{\gamma}_q$-correct at t_q on remote node q, then $\mathbf{R}_{p,q} = \mathbf{Q}_{p,q} \cdot \mathbf{R}_q$ is $\boldsymbol{\gamma}_{p,q}$-correct at t'_p, whereby*

$$\boldsymbol{\gamma}_{p,q} = \mathbf{Q}_{p,q} \cdot \boldsymbol{\gamma}_q$$

and no resynchronizations occurred in between.

Proof Similar to the proof of Lemma 9 by starting out with $\omega(t_q) \in v_q(t_q)(1+\boldsymbol{\gamma}_q)$, replacing $v_q(t_q)$ by $v_p(t'_p)$ from (26), and noting that $\omega(t'_p) = \omega(t_q)$. □

4 Clock Rate Algorithm

In this section we stitch together an algorithm, called CRA, for both external and internal clock rate synchronization based on the elements of Section 3. For ease of presentation, we first assume a fault-free system of $n \geq 2$ nodes, connected by a communication subsystem (Assumption 4 and 5), where each node p hosts a processor (Assumption 3) and a clock $\mathcal{C}_p$ driven by an oscillator $\mathcal{O}_p$ (Assumption 1 and 2). Our starting point is the round structure of the algorithm, followed by the development of the pieces for external and then internal rate synchronization. Faults are treated generically, so that only a few specific modifications of the algorithm are necessary to work correctly upon an established fault model. A simple example wraps up this section, providing a good insight how the algorithm works.

4.1 Round Structure

First we lay out the structure of our algorithm, which is based on rounds as known from other distributed algorithms. Periodically, every *rate resynchronization period* P_{CRA}, each node executes the same algorithm, which consists roughly of mutual rate measurements and the computation of a suitable rate correction. The rounds are a product of a coexistent clock state algorithm with *state resynchronization period* P_{CSA}, whereby global precision $\Pi_{\max}$ doesn't need to be too small to make our algorithm working. In fact, a round-less version is also conceivable by an adaptation of the value of $\Pi_{\max}$.

Let us examine a particular round k by using Figure 4. Logically it starts at $kP_{\text{CRA}} + D$ and ends at $(k+1)P_{\text{CRA}} + D$ lasting a duration of P_{CRA} seconds. The rates of the clocks are going to be actively influenced by our algorithm at these points in time symbolized by SYN_k and SYN_{k+1}. During a round, we need to carry out the protocol of Figure 3 in order to make relative rate measurements. More specifically, shortly after the beginning of round k, we initiate a *full message exchange* FME'_k for the first messages, and near the end of the round the second messages belonging to FME_{k+1}. The reason to have two separate FMEs instead of using just the ones near at the end of each round back-to-back is, that we need to exclude any rate synchronization during rate measurement periods. Intervening state synchronizations are allowed to occur, since they can easily be taken care by summing up all state corrections. Indeed, the additional FMEs at the beginning of each round can be piggy-backed on the FMEs from state synchronization.

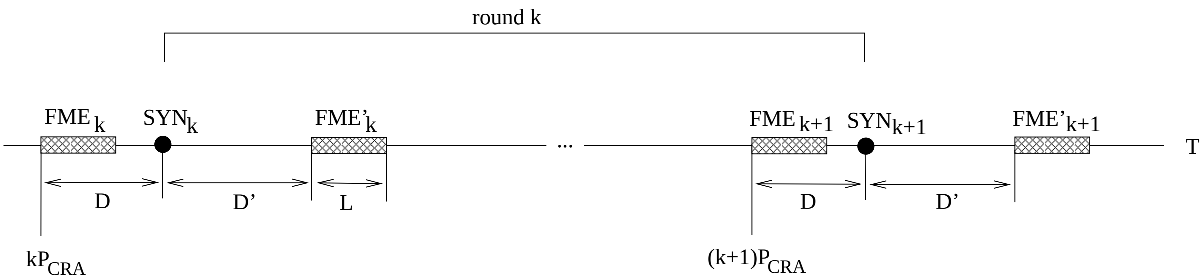

Figure 4: Execution of a Round

To make our algorithm running properly, we have to set up large enough delays to account for the longest possible duration of any FME and any computation inside algorithm CRA. In particular, delay D consists of the worst case FME duration L and the worst case computation duration E. Delay D' lessens the active rate measurement duration to $P_{\text{CRA}} - D - D'$. The following lemma helps us to assign D and D'.

Lemma 10 (FME and Computation Duration) *Complying to Assumptions 1-5, any FME is completed within*

$$L = (1 + 2\rho_{\max}) \left(\frac{\Pi_{\max}}{1 - 2\rho_{\max}} + \tau_{\max} + \lambda_{\max} + \Delta t_{\max} + \epsilon_{\max} \right)$$

and any single computation for the clock rate synchronization within

$$E = (1 + 2\rho_{\max})\eta_{\max}$$

logical seconds.

Proof Suppose $\mathcal{C}_p$ is the fastest and $\mathcal{C}_q$ the slowest clock in our ensemble. Let t^I be the real-time when node p initiates as first one its broadcast of the particular FME. From Assumption 2 we know that $C_q(t^I) \geq C_p(t^I) - \Pi_{\max}$, hence node q takes no more then $\pi_q \leq \Pi_{\max}/(1 - 2\rho_q)$ seconds to initiate its broadcast as the latest participant of the same FME. This follows from Lemma 2 and condition (2), whereby the usage of the maximum oscillator drift includes the deviation from the stability. Before the message from node q gets transmitted, it can experiences a maximum broadcast operation delay $\tau_{\max}$ and a maximum broadcast latency $\lambda_{\max}$. Furthermore, the transmission can take as long as $\Delta t_{\max} + \epsilon_{\max}$ seconds to reach its receiver. Since we are interested in the longest logical duration L of an FME, we map the sum of expired real-times to the fastest clock p, hence $L = C_p(t^I + \pi_q + \lambda_{\max} + \tau_{\max} + \Delta t_{\max} + \epsilon_{\max}) - C_p(t^I)$. A following application of Lemma 2 and condition (2) yields L.

From Assumption 3 we know that any computation doesn't take longer than $\eta_{\max}$ seconds. Once again, mapping this duration onto the fastest clock, we get the desired result. $\square$

Given these logical duration both for FME and computation, we are able to quantify the delay D in our round structure as shown in Figure 4. Obviously, D has to be at least $L + E$, hence we set

$$D = (1 + 2\rho_{\max}) \left(\frac{\Pi_{\max}}{1 - 2\rho_{\max}} + \tau_{\max} + \lambda_{\max} + \Delta t_{\max} + \epsilon_{\max} + \eta_{\max} \right). \tag{28}$$

Setting

$$D' = P_{\text{CSA}} - D \tag{29}$$

for practical reasons, we have to choose the rate resynchronization period P_{CRA} larger than $2P_{\text{CSA}}$, since it has to be at least $2(D + D')$. Obviously, P_{CSA} has to be at least D.

4.2 External Clock Rate Algorithm

After carrying out FME'$_k$ and FME$_{k+1}$ during round k, node p is able to build up a set of quotient rate intervals $\boldsymbol{\mathcal{Q}}_p = \{\mathbf{Q}_{p,1}, \ldots, \mathbf{Q}_{p,n}\}$ by using the formula from Lemma 8. A straightforward approach would apply a particular point-based convergence function, for instance FTA or FTM as proposed in [DLPSW83], upon the associated reference points to obtain a new one for adjusting the rate at SYN$_{k+1}$. This sounds reasonable, because local clocks would get in sync with a majority, but there are no means of external rate synchronization.

However, we can take advantage of local rate intervals, which provide vital information about clock rate. Pointing out that if clock $\mathcal{C}_p$ has rate v_p, then the ideal rate of 1 with respect to the current rate is merely the reciprocal of it. Therefore knowing a range of $1/v_p$ enables clock $\mathcal{C}_p$ to adjust it's rate accordingly. The crux is to narrow down this range with the help of remote clocks, where the remote rate intervals are carrier of this information. From Lemma 9 we know that each correct remote rate interval of the set $\boldsymbol{\mathcal{R}}_p = \{\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n}\}$ computed by node p contains the searched value of $1/v_p$.

Considering the dynamic nature of our rate intervals, we have to make them compatible with each other in a meaningful way. More specifically, the broadcasted local rate interval $\mathbf{R}_q$ at FME'$_k$ of node q stems from SYN$_k$, so we need to compensate for $D' + L$. This happens with the formula from Lemma 3, whereby D' is necessary to bridge the time towards the initiation of FME'$_k$ and L for its duration. Afterwards we can obtain the remote rate interval, which needs to be deteriorated by D at node p to finally reach SYN$_{k+1}$.

Having the set of compatible remote rate intervals $\boldsymbol{\mathcal{R}}_p$ at hand, a feature for external rate synchronization needs to be provided. We seize the idea of *clock rate validation* rooted in [Sch94], which works basically as follows: Suppose there are a few *primary nodes* in our system that act as rate references, e.g. their clocks are disciplined by GPS receivers, cf. [Dan97]. Their contributing remote rate intervals will be of short lengths due to their good rate knowledge. On the other hand, the remote rate intervals from nodes within the ensemble exhibit usually longer lengths, but are of larger cardinality. Since all intervals are supposed to include $1/v_p$, rate validation can be viewed abstractly as a selection function $\mathcal{VAL}_{\mathcal{F}}(\cdot)$ defined in the following way.

Definition 8 (Ideal Validation) *Let $\boldsymbol{\mathcal{R}}'_p$ resp. $\boldsymbol{\mathcal{R}}''_p$ be a set of remote rate intervals at node p from primary nodes resp. from nodes of the ensemble according to fault model $\mathcal{F}$. A validation function $\mathcal{VAL}_{\mathcal{F}}(\boldsymbol{\mathcal{R}}' \cup \boldsymbol{\mathcal{R}}'')$ is said to be ideal, iff it either outputs the subset of correct onces of $\boldsymbol{\mathcal{R}}'$ providing it is non-empty, or $\boldsymbol{\mathcal{R}}''$ otherwise.*

The actual mechanism to carry out clock rate validation is not relevant at the moment (see forthcoming paper []), but this kind of ideal validation isolates us from the sublte issues about toggling between internal and external rate synchronization. Even more, it is essential to recognize that the length of the rate intervals are of major interest for external rate synchronization due to the implicit inclusion of $1/v_p$ in case of correct ones. The accompaning reference points play only a minor role in terms of external rate synchronization.

Given the set of validated remote rate intervals, a suitable interval-based *convergence function* $\mathcal{CV}_{\mathcal{F}}(\cdot)$ tailored to fault model $\mathcal{F}$ is in charge of computing a new interval $\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}}$ that encloses $1/v_p$. However, instead of dealing with concrete convergence functions $\mathcal{CV}_{\mathcal{F}}(\cdot)$, we pin-point abstract properties of them for the external rate algorithm, cf. [Sch87]. First of all, we require them to be translation invariant and weakly monotonic in the following sense.

Definition 9 (Translation Invariance, Weak Monotonicity) *Given two sets $\boldsymbol{\mathcal{I}} = \{\mathbf{I}_1, \ldots, \mathbf{I}_n\}$ and $\boldsymbol{\mathcal{J}} = \{\mathbf{J}_1, \ldots, \mathbf{J}_n\}$ of $n \geq 1$ intervals. An interval-valued function $\mathbf{f}(\cdot)$ is called* translation invariant *iff $\mathbf{f}(\mathbf{I}_1 + y, \ldots, \mathbf{I}_n + y) = \mathbf{f}(\mathbf{I}_1, \ldots, \mathbf{I}_n) + y$ for any real y, and* weakly monotonic *iff $\mathbf{I}_i \subseteq \mathbf{J}_i$ with $ref(\mathbf{I}_i) = ref(\mathbf{J}_i)$ for all $1 \leq i \leq n$ implies $\mathbf{f}(\boldsymbol{\mathcal{I}}) \subseteq \mathbf{f}(\boldsymbol{\mathcal{J}})$.*

As already pointed out, the length of the remote rate intervals are important for external rate synchronization purpose. More specifically, a *drift preservation function* $\mathcal{DP}(\cdot)$ relates the length of the fed-in remote rate intervals with the computed one. Section 5 will treat these issues in greater details when analyzing the algorithm.

Definition 10 (Drift Preservation) *Let $\boldsymbol{\mathcal{R}}_p = \{\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n}\}$ be a set of remote rate intervals in accordance with fault model $\mathcal{F}$, and let $\mathbf{V}_{p,i}$ bounds such that $align(\mathbf{R}_{p,i}) \subseteq \mathbf{V}_{p,i}$ for correct $\mathbf{R}_{p,i}$. A convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ is characterized by a weakly monotonic drift preservation function $\mathcal{DP}(\cdot)$ iff*

$$align(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}}) \subseteq \mathcal{DP}(\mathbf{V}_{p,1}, \ldots, \mathbf{V}_{p,n}; \ldots),$$

where $\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}} = \mathcal{CV}_{\mathcal{F}}(\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n})$.

It remains to explain how the algorithm accomplishes the rate adjustment according to $\mathbf{R}_p^{\mathcal{CV}_\mathcal{F}}$. In particular, we need to determine a new factor S'_p for the oscillator-clock coupling and a new local rate interval $\mathbf{R}'_p$. The first issue is straightforward, since we know that $1 \in v_p \mathbf{R}_p^{\mathcal{CV}_\mathcal{F}}$ at SYN_{k+1}. If $r_p^{\mathcal{CV}_\mathcal{F}} = ref(\mathbf{R}_p^{\mathcal{CV}_\mathcal{F}})$, then our best approximation for the clock rate is given by $v_p = 1/r_p^{\mathcal{CV}_\mathcal{F}}$, hence we set

$$S'_p = S_p r_p^{\mathcal{CV}_\mathcal{F}} \tag{30}$$

to enforce the new rate. Additionally we check for $|S'_p f_p - 1| \leq \rho_p$ to ensure a feasible rate adjustment, otherwise the clock is declared as faulty. The second issue deals with the computation of the rate interval $\mathbf{R}'_p$ in respect to the new rate $v'_p = v_p r_p^{\mathcal{CV}_\mathcal{F}}$. Because of $1 \in v_p \mathbf{R}_p^{\mathcal{CV}_\mathcal{F}} = (v'_p / r_p^{\mathcal{CV}_\mathcal{F}}) \mathbf{R}_p^{\mathcal{CV}_\mathcal{F}}$ we can set

$$\mathbf{R}'_p = \frac{1}{r_p^{\mathcal{CV}_\mathcal{F}}} \mathbf{R}_p^{\mathcal{CV}_\mathcal{F}}. \tag{31}$$

Now we have developed and justified all parts of algorithm CRA. Before diving into the analysis, we give a crisp summary in Algorithm 1. Also take a look at Figure 4 for an easier understanding. The employed convergence function has to be translation invariant, weakly monotonic and characterized by a drift preservation function in regard of fault-model $\mathcal{F}$; the employed validation function is supposed to be ideal.

Algorithm 1 (CRA) *Complying to Assumptions 1-5, each node p performs the following actions:*

#1	*(FME'$_k$)*	*at* $kP_{\mathrm{CRA}} + D + D'$ *initiate broadcast containing the local rate interval* $\mathbf{R}_p$ *from* SYN_k *and timestamp it with* T_p
#2		*until* $kP_{\mathrm{CRA}} + D + D' + L$ *receive messages from remote nodes q and timestamp them with* $T_{p,q}$
#3	*(FME$_{k+1}$)*	*at* $(k+1)P_{\mathrm{CRA}}$ *initiate broadcast containing the sum* U_p *of applied state synchronizations during round k and timestamp it with* T'_p
#4		*until* $(k+1)P_{\mathrm{CRA}} + L$ *receive messages from nodes q and timestamp them with* $T'_{p,q}$
#5	*(COMP$_{k+1}$)*	*at* $(k+1)P_{\mathrm{CRA}} + L$ *compute the set of quotient rate intervals* $\boldsymbol{\mathcal{Q}}_p$*, whereby* $\mathbf{Q}_{p,q} \leftarrow \left[\frac{T'_q - T_q + U_q}{T'_{p,q} - T_{p,q} + U_p} \pm \left(\frac{(\sigma_p + \sigma_q)(T'_q - T_q + U_q)}{2(1-2\rho_q)} + \frac{\epsilon_{\max}(1+2\rho_p)}{T'_{p,q} - T_{p,q} + U_p}\right)\right]$ *for* $q \neq p$
#6		*compute the set of remote rate intervals* $\boldsymbol{\mathcal{R}}_p$*, whereby* $\mathbf{R}_{p,q} \leftarrow \left(\mathbf{R}_q + \left[0 \pm \frac{\sigma_q(D'+L)}{1-2\rho_q}\right]\right) \cdot \mathbf{Q}_{p,q} + \left[0 \pm ref(\mathbf{Q}_{p,q}) \frac{\sigma_p D}{1-2\rho_p}\right]$ *for* $q \neq p$*, and* $\mathbf{R}_{p,p} \leftarrow \mathbf{R}_p + \left[0 \pm \frac{\sigma_p P_{\mathrm{CRA}}}{1-2\rho_p}\right]$
#7		*invoke rate validation to get* $\{\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n}\} \leftarrow \mathcal{VAL}_\mathcal{F}(\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n})$
#8		*apply convergence function to get* $\mathbf{R}_p^{\mathcal{CV}_\mathcal{F}} \leftarrow \mathcal{CV}_\mathcal{F}(\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n})$
#9	*(SYN$_{k+1}$)*	*at* $(k+1)P_{\mathrm{CRA}} + D$ *adjust clock rate by setting* $S_p \leftarrow S_p \cdot ref(\mathbf{R}_p^{\mathcal{CV}_\mathcal{F}})$
#10		*reset local rate interval by* $\mathbf{R}_p \leftarrow \mathbf{R}_p^{\mathcal{CV}_\mathcal{F}} / ref(\mathbf{R}_p^{\mathcal{CV}_\mathcal{F}})$

Actions #1 to #4 perform the protocol to carry out the relative rate measurement from Figure 3, resulting in the calculation of the quotient rate intervals in #5 by virtue of Lemma 8. Action #6 transforms the received rate intervals into remote rate intervals according to Lemma 3 and 9, which are subsequently fed into the validation function in action #7 and the convergence function in action #8. The computed interval is used to adjust instantaneously the clock rate in action #9 and to update the local rate interval in #10.

The above algorithm works for $k \geq 1$. In the initial case $k = 0$, we begin with local rate interval $\mathbf{R}_p = [\rho_p/(1+\rho_p), 1, \rho_p/(1-\rho_p)]$ and coupling factor $S_p = 1/f_p$ as justified by Lemma 1. The results of analyzing algorithm CRA, in particular the run of drifts, will be given in Theorem 2.

4.3 Internal Clock Rate Algorithm

As long as remote rate intervals from primary nodes are passed through by the clock rate validation $\mathcal{VAL}_{\mathcal{F}}(\cdot)$, they determine interval $\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}}$ by virtue of convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$. As a consequence, the rates of the clocks in our ensemble are disciplined by external clocks. In this operation mode, we say the algorithm performs external rate synchronization, resulting in a small maximum drift δ and consequently a small consonance γ according to Corollary 1.

However, in case that remote rate intervals from external references get rejected by the clock rate validation or are absent altogether, we have to rely on the remote rate intervals from the other nodes of the ensemble. They are generally longer and their reference points have a tendency to drift apart. Thus, it is the purpose of the convergence function to keep them together, whereby their lengths are still growing. This operation mode is known as internal rate synchronization, ensuing a bounded consonance in spite of an increasing maximum drift.

To understand the preservation of the consonance, we recall the notion of internal global rate $\omega(t)$ and $\boldsymbol{\gamma}$-correctness. Suppose that each local rate interval belonging to non-faulty clock is $\boldsymbol{\gamma}_0$-correct at SYN_k, the beginning of round k. During the round we have to deteriorate it in order to compensate for the stability of the local clock. This is done by Lemma 6 ending up with $\boldsymbol{\gamma}$-correct local rate intervals at SYN_{k+1}. Note that these interval properties are merely of artificial nature, since the algorithm has no explicit handle on them.

Just enlarging $\boldsymbol{\gamma}_0$ to $\boldsymbol{\gamma}$ doesn't guarantee a bounded consonance for any $t \to \infty$. Therefore, at the end of round k the clock rates have to be manipulated in such a way, that the ensuing local rate intervals become $\boldsymbol{\gamma}_0$-consonant, demanding $\boldsymbol{\gamma}_0 \subset \boldsymbol{\gamma}$. Observe carefully, that we cannot safely assert $\boldsymbol{\gamma}_0$-correctness here, since it may be the case that the internal global rate ω^k from round k does not fit into the new $\boldsymbol{\gamma}_0$ consonance intervals. Defining a suitable new internal global rate ω^{k+1} resolves this deficiency and reassures $\boldsymbol{\gamma}_0$-correctness. As a result, our introduced internal global rate makes discrete leaps at resynchronization instants and remains constant otherwise, cf. Figure 2.

If we impose further properties on the convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$, algorithm CRA serves not only external but internal rate synchronization as well. For that purpose imagine two different local nodes p and q that are about to measure their rates in cooperation with another remote node i. After the execution of one round, the local rate interval $\mathbf{R}_i$ at node i leaves two remote rate intervals $\mathbf{R}_{p,i}$ resp. $\mathbf{R}_{q,i}$ at node p resp. q. The iteration with all nodes $1 \leq i \leq n$ results in the sets $\boldsymbol{\mathcal{R}}_p = \{\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n}\}$ and $\boldsymbol{\mathcal{R}}_q = \{\mathbf{R}_{q,1}, \ldots, \mathbf{R}_{q,n}\}$. In consideration of our assumptions, we are able to show in Section 5.1 that non-faulty remote rate intervals w.r.t fault model $\mathcal{F}$ have the following properties:

- $\boldsymbol{\gamma}_{p,i}$-correctness of $\mathbf{R}_{p,i}$,
- $\boldsymbol{\gamma}_{q,i}$-correctness of $\mathbf{R}_{q,i}$,
- $\boldsymbol{\gamma}^i$-correctness of $\mathbf{R}_{p,i}$ and $\mathbf{R}_{q,i}$,
- $\boldsymbol{\gamma}_H$-correctness of any non-faulty remote rate interval,
- $\boldsymbol{\gamma}_I$-consonance of any set $\{\mathbf{R}_{p,i}, \mathbf{R}_{q,i}\}$ with $||\boldsymbol{\gamma}_I|| < ||\boldsymbol{\gamma}_H||$.

Applying convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ at node p resp. q yields $\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}} = \mathcal{CV}_{\mathcal{F}}(\mathbf{R}_{p,1}, \ldots, \mathbf{R}_{p,n})$ resp. $\mathbf{R}_q^{\mathcal{CV}_{\mathcal{F}}} = \mathcal{CV}_{\mathcal{F}}(\mathbf{R}_{q,1}, \ldots, \mathbf{R}_{q,n})$, that has to both preserve and enhance consonance. This will be specified with the help of a *consonance preservation function* $\mathcal{CP}(\cdot)$ and a *consonance enhancement function* $\mathcal{CE}(\cdot)$ in the following way.

Definition 11 (Consonance Preservation) *Given two sets of remote rate intervals $\boldsymbol{\mathcal{R}}_p$ and $\boldsymbol{\mathcal{R}}_q$ with the above made preconditions. A convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ is characterized by a weakly monotonic consonance preservation function $\mathcal{CP}(\cdot)$ iff*

$$\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}} \textit{ is } \mathcal{CP}(\boldsymbol{\gamma}_{p,1}\ldots,\boldsymbol{\gamma}_{p,n};\boldsymbol{\gamma}_H;\boldsymbol{\gamma}_I;\ldots)-\textit{correct and } \mathbf{R}_q^{\mathcal{CV}_{\mathcal{F}}} \textit{ is } \mathcal{CP}(\boldsymbol{\gamma}_{q,1}\ldots,\boldsymbol{\gamma}_{q,n};\boldsymbol{\gamma}_H;\boldsymbol{\gamma}_I;\ldots)-\textit{correct}$$

with $||\mathcal{CP}(\boldsymbol{\gamma}^1\ldots,\boldsymbol{\gamma}^n;\boldsymbol{\gamma}_H;\boldsymbol{\gamma}_I;\ldots)|| = \mathcal{O}(||\boldsymbol{\gamma}_H||)$.

Definition 12 (Consonance Enhancement) *Given two sets of remote rate intervals $\boldsymbol{\mathcal{R}}_p$ and $\boldsymbol{\mathcal{R}}_q$ with the above made preconditions. A convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ is characterized by a weakly monotonic consonance enhancement function $\mathcal{CE}(\cdot)$ iff*

$$\{\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}},\mathbf{R}_q^{\mathcal{CV}_{\mathcal{F}}}\} \textit{ is } \boldsymbol{\gamma}^{\mathcal{CV}_{\mathcal{F}}} - \textit{consonant satisfying } ||\boldsymbol{\gamma}^{\mathcal{CV}_{\mathcal{F}}}|| = \mathcal{CE}(\boldsymbol{\gamma}^1\ldots,\boldsymbol{\gamma}^n;\boldsymbol{\gamma}_H;\boldsymbol{\gamma}_I;\ldots)$$

with $\mathcal{CE}(\boldsymbol{\gamma}^1\ldots,\boldsymbol{\gamma}^n;\boldsymbol{\gamma}_H;\boldsymbol{\gamma}_I;\ldots) < ||\boldsymbol{\gamma}_H||$.

The results of analyzing algorithm CRA, in particular the bounded consonance, will be summarized in Theorem 1.

4.4 Fault Model

So far we have sidestepped the issue of faults in our system, which allowed us to establish a generic framework for clock rate synchronization. Only the validation function $\mathcal{VAL}_{\mathcal{F}}(\cdot)$ and the convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ need to be modified in respect to fault model $\mathcal{F}$. In the following we recap their duties: A validation function has to mask out faults stemming from primary nodes by making use of rate information from the other nodes of the ensemble. Ideally, it either outputs the subset of correct rate interval from primary nodes or the set of rate intervals from the others, as given in Definition 8. A fault model $\mathcal{F}$ is needed to specify the potential faults in terms of plugged in rate intervals, so that a suitable validation function $\mathcal{VAL}_{\mathcal{F}}(\cdot)$ can be applied. A convergence function has to produce a rate interval that maintains both drift (Definition 10) and consonance (Definition 11 and 12) in spite of faults stemming from nodes of the ensemble. Again, they need to be characterized by a fault model $\mathcal{F}$, so that an appropriate convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ can be utilized. Both functions will call for certain fault-tolerance requirements, e.g. fraction of non-faulty primary or ensemble nodes.

It remains to illustrate how system faults lead to faulty rate intervals. They can be faulty in terms of their reference points and/or their lengths, resulting in a catalogue from which a specific fault model $\mathcal{F}$ can be built, cf. [AK96]. For that purpose, imagine a broadcasting node and two receiving nodes, carrying out the rate measurement protocols (two FMEs) and the preprocessing steps to obtain their remote rate intervals for a particular round. In order to understand the variety of faults, it is advantageous to distinguish between two perspectives:

- Viewed from the perspective of a single receiving node, a remote rate interval can experience
  - *omission faults*, caused by an omissive broadcasting node or transient errors during message reception, making it to an empty interval $\emptyset$.
  - *timing faults*, caused by a faulty broadcasting node or excessive transmission delays, rendering it as non-correct and/or non-$\boldsymbol{\gamma}$-correct.
  - *value faults*, caused by a faulty broadcasting node or a damaged message, rendering it as non-correct and/or non-$\boldsymbol{\gamma}$-correct. In contrast to accuracy intervals as used in clock state synchronization, rate intervals have an easy to check meaningful lengths, so we don't need to consider truncated, bounded or even unbounded intervals, cf. [Mar90].

- Viewed from the perspective of corresponding receiving nodes, their remote rate intervals can be entagled with
  - *crash faults*, which are consistently detectable. However, a node that crashes during a broadcast operation might produce inconsistent receptions.
  - *omission faults*, which are usually differently at different receiving nodes. Traditionally, they are attributed to the broadcasting node although most receive omissions occur independently.
  - *restricted faults*, which are consistent and can be tolerated with moderate efforts.
  - *arbitrary faults*, which are inconsistent including the byzantine case. They may be caused by nodes sending different messages to different receivers or by excessive transmission delays at receiving nodes. Such faults can also occur in broadcast-type networks, since the broadcast operation is not assumed to be reliable. However, we rule out the possibility of impersonating other nodes or jamming the network.

It is beyond the scope of this paper to set up fault models along with suitable validation and convergence functions. A forthcoming paper deals with these fundamental issues, cf. [Sch97].

4.5 Example

A simple execution should demonstrate algorithm CRA. We assume three nodes with stable clocks ($\sigma_{\max} = 0$ *ppm/sec*) having rates $v_1 = 0.8$ *Sec/sec*, $v_2 = 0.9$ *Sec/sec* and $v_3 = 1.3$ *Sec/sec*, hence the drift of our ensemble is 0.3 *Sec/sec* and the consonance is 0.5 *Sec/sec*. Of course, the clock rates are not directly observable, but local rate intervals $\mathbf{R}_1 = [0.15, 1, 0.3]$, $\mathbf{R}_2 = [0.2, 1, 0.4]$ and $\mathbf{R}_3 = [0.3, 1, 0]$ capture them by fulfilling condition $1 \in v_i \mathbf{R}_i$ for all $1 \leq i \leq 3$ from Definition 4.

The protocol for relative rate measurement yields the quotient rate intervals $\mathbf{Q}_{p,q} = v_q/v_p$ for all $1 \leq p, q \leq 3$ when neglecting transmission uncertainties ($\epsilon_{\max} = 0$ *sec*). In addition local rate intervals are exchanged and subsequently converted into remote rate intervals by use of $\mathbf{R}_{p,q} = \mathbf{Q}_{p,q} \cdot \mathbf{R}_q$ from Lemma 9. The resulting matrix of remote rate intervals reads

$$\mathbf{R} = \begin{pmatrix} [0.15, 1, 0.3] & [0.225, 1.125, 0.45] & [0.488, 1.625, 0] \\ [0.133, 0.888, 0.266] & [0.2, 1, 0.4] & [0.433, 1.444, 0] \\ [0.092, 0.615, 0.185] & [0.138, 0.692, 0.277] & [0.3, 1, 0] \end{pmatrix}.$$

Only local information was used to calculate the remote rate intervals. Figure 5 depicts them from a global perspective, where each dashed block represents the local view of a node. Qualitatively we can say that nodes have a similar view up to a particular shift (nodes with faster clocks to the left) and stretch (nodes with slower clocks possess larger intervals).

Let's take a closer look of node 1. All it's remote rate intervals include the desired reciprocal rate $1/v_1 = 1.25$ *sec/Sec*. Using plain intersection as convergence function and setting the reference point in the middle, we get $\mathbf{R}_1^{\mathcal{CV}_{\mathcal{F}}} = [0.082, 1.218, 0.082]$. For adjusting the clock rate we alter multiplicatively the oscillator-clock coupling by 1.218 leading to the new rate $v_1' = 0.9744$ *Sec/sec*. Moreover the new local rate interval in respect to the changed rate becomes to $\mathbf{R}_1' = [0.067, 1, 0.067]$, which again goes hand in hand with Definition 4.

Analogous results hold for the other nodes. In particular, the rate adjustment factors are 1.083 for node 2 and 0.75 for node 3, hence the new rates turn out to be $v_2' = 0.9747$ *Sec/sec* and $v_3' = 0.975$ *Sec/sec*, respectively. The new local rate intervals $\mathbf{R}_2'$ and $\mathbf{R}_3'$ happens to be identical with $\mathbf{R}_1'$ due to our trivial scenario. Suppose all rates were adjusted that way, we can reckon a drift

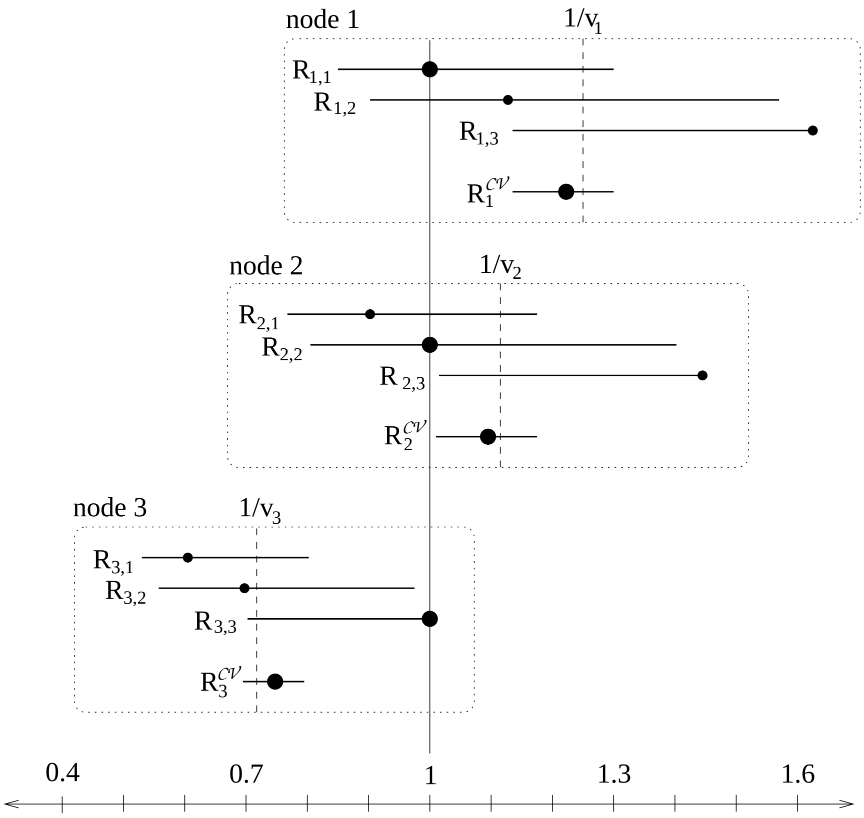

Figure 5: Global view of remote rate intervals

$\delta = 0.0255$ Sec/sec and a consonance $\gamma = 0.00056$ Sec/sec. Not surprisingly, this is an excellent result, but we should put it in perspective with the idealized assumptions.

5 Analysis

The analysis of Algorithm CRA comes in four subsections concerning the dissemination of rate and consonance intervals within a single round, the applied rate adjustments, the consonance (internal rate synchronization) and drift (external rate synchronization) and of clocks. A terminating section shows how clock rate and state synchronization work together.

5.1 Interval Dissemination

At the very first, we have to agree upon the real-time, when a particular round starts, since clocks don't resynchronize simulatenously. So far we have used SYN_k to symbolize the begin of round k, whereas the corresponding real-time of a non-faulty node p is t_p^k. Following the line of arguments from the proof of Lemma 10 and relying on the global precision $\Pi_{\max}$ from Assumption 2, it is easy to show that

$$\left|t_p^k - t_q^k\right| \leq \frac{1+2\rho_{\max}}{1-2\rho_{\max}}\Pi_{\max}. \tag{32}$$

where p and q are non-faulty nodes.

However, rate intervals (Definition 4) and consonance intervals (Definition 5) are expressed over a common point of time, so the maximum of the resynchronization times t_p^k among non-faulty nodes p can be exclaimed as the beginning of round k, denoted by t^k. If clocks are stable, the clock rate $v_p(t_p^k)$ at t_p^k remains constant until t^k. Fortunately, since $\Pi_{\max}$ is assumed to be very small compared to P_{CRA}, we can justify the approximation

$$v_p(t^k) = v_p(t_p^k) + \mathcal{O}(\sigma_p \Pi_{\max}) \tag{33}$$

by combining (12) and (32). Unlike to clock state synchronization, it is our goal to collapse the period of resychronization into a single point of time. Note that if $\Pi_{\max}$ is considerably large, we cannot make this kind of simplification. The following Definition smoothes the way for rate intervals and consonance intervals along with internal global rate having their origin at t^k for round $k \geq 0$.

Definition 13 (Round Start) *Let t_p^k be the resynchronization times for non-faulty nodes p, when switching to round $k \geq 0$ takes place. The begin of round k is determined by $t^k = max\{t_p^k\}$ among non-faulty nodes p. If rate interval $\mathbf{R}_p(t_p^k)$ is correct and $\bigcap v_p(t_p^k)(ref(\mathbf{R}_p(t_p^k)) + \boldsymbol{\gamma}^k) \neq \emptyset$ for non-faulty nodes p holds, then we assert rate interval $\mathbf{R}_p(t^k) = \mathbf{R}_p(t_p^k) + [0 \pm \mathcal{O}(\sigma_p \Pi_{\max})]$ as*

(1) correct, and

(2) $\Big(\boldsymbol{\gamma}^k + [0 \pm \mathcal{O}(\sigma_{\max}\Pi_{\max})]\Big)$-correct

w.r.t internal global rate $\omega(t^k) \in \bigcap_{\mathrm{non-faulty}\, p} v_p(t_p^k)(ref(\mathbf{R}_p(t_p^k)) + \boldsymbol{\gamma}^k)$.

Three remarks are important to make about internal global rate: First, the basis to define $\omega(t^k)$ was a set of $\boldsymbol{\gamma}^k$-consonance rate intervals at a common point of time, cf. Definition 6. We relaxed it for different times and appended $\mathcal{O}(\sigma_{\max}\Pi_{\max})$ to $\boldsymbol{\gamma}^k$. Second, note that any value in the above intersection can be chosen as $\omega(t^k)$. Finally, internal global rate remains constant during a round, thus $\omega(t) = \omega^k = \omega(t^k)$ for $t^k \leq t < t^{k+1}$. As a consequence, $\omega(t)$ is a piecewise constant function as depicted in Figure 2.

During each round the local rate interval $\mathbf{R}_q$ of a remote node q will be transferred to the rate interval $\mathbf{R}_{p,q}$ at local node p with the help of two FMEs. Since the computation of a remote rate interval involves the multiplication of the associated quotient rate interval $\mathbf{Q}_{p,q}$, we begin with obtaining properties on them. The final results are given in Lemma 13 and 14.

Lemma 11 (Quotient Rate Interval Bounds) *If $\mathbf{R}_q$ is a correct local rate interval at remote node q when a round commences, then the quotient rate interval $\mathbf{Q}_{p,q}$ at local node $p \neq q$ has the following properties:*

(1) $align(\mathbf{Q}_{p,q}) \subseteq \left[0 \pm \left(\frac{(\sigma_p+\sigma_q)P_{\text{CRA}}}{2(1-2\rho_q)} + \frac{\epsilon_{\max}}{(P_{\text{CRA}}-P_{\text{CSA}}-B)(1-10\rho_{\max})}\right)\right] + \left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\text{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\text{CRA}}^2} + \sigma_{\max}^2 P_{\text{CRA}}^2 + \sigma_{\max}\epsilon_{\max}\right)\right]$

(2) $ref(\mathbf{Q}_{p,q}) \in swap(\mathbf{R}_q) + \left[0 \pm \left(2\rho_p + \frac{\sigma_q P_{\text{CRA}}}{1-2\rho_q} + \frac{2B+\epsilon_{\max}}{P_{\text{CRA}}-P_{\text{CSA}}}\right)\right] + \left[0 \pm \mathcal{O}\left(\frac{(B+\epsilon_{\max})\rho_{\max}}{P_{\text{CRA}}} + \frac{(B+\epsilon_{\max})^2}{P_{\text{CRA}}^2} + (\rho_{\max} + \sigma_{\max}P_{\text{CRA}})^2\right)\right]$

with maximum logical broadcast delay $B = (1+2\rho_{\max})(\tau_{\max}+\lambda_{\max})$.

Proof Looking at the formula for $\mathbf{Q}_{p,q}$ at action #5 in Algorithm 1, our first step is devoted to find bounds on the duration $\Delta T_q = T'_q - T_q$ and the corresponding duration $\Delta T_{p,q} = T'_{p,q} - T_{p,q}$. The accumulated state corrections U_p and U_q can be ignored, since we assume a transparent state synchronization.

To get a handle on $T'_q - T_q$, we have to figure out the range of sending timestamps belonging to FME'$_k$ and FME$_{k+1}$ at remote node q during an arbitrary round k. Directly from Algorithm 1 and consulting the proof of Lemma 10, we see that $0 \leq T_q - (kP_{\text{CRA}} + D + D') \leq B$ and $0 \leq T'_q - (k+1)P_{\text{CRA}} \leq B$. Adding these unequation up and considering setting (29), we end up with

$$|\Delta T_q - (P_{\text{CRA}} - P_{\text{CSA}})| \leq B. \tag{34}$$

Carrying over the bounds on ΔT_q to the real-time counterpart Δt_q and further to $\Delta t_{p,q}$ at node p will finally give us bounds on $\Delta T_{p,q}$. If $\mathbf{R}_q = [\theta_q^-, 1, \theta_q^+]$ is correct at t^k, then a deterioration by P_{CRA} according to Lemma 3 makes it correct throughout round k. Therefore it follows by using the asymptotic approximation $(1 \pm x)^{-1} = 1 \mp x + \mathcal{O}(x^2)$ for $x \to 0$ that the clock rate $v_q(t)$ is bounded by

$$v_q(t) \leq 1 + \theta_q^- + \frac{\sigma_q P_{\text{CRA}}}{1-2\rho_q} + \mathcal{O}\left((||\mathbf{R}_q|| + \sigma_q P_{\text{CRA}})^2\right) \tag{35}$$

and

$$v_q(t) \geq 1 - \theta_q^+ - \frac{\sigma_q P_{\text{CRA}}}{1-2\rho_q} - \mathcal{O}\left((||\mathbf{R}_q|| + \sigma_q P_{\text{CRA}})^2\right) \tag{36}$$

for all $t \in [t^k, t^{k+1}]$. From Lemma 7 we can obtain the relation

$$\frac{\Delta T_q}{\Delta t_q}(1 - \frac{\sigma_q}{2}\Delta t_q) - \mathcal{O}(\sigma_q^2 \Delta t_q^2) \leq v_q(t) \leq \frac{\Delta T_q}{\Delta t_q}(1 + \frac{\sigma_q}{2}\Delta t_q) + \mathcal{O}(\sigma_q^2 \Delta t_q^2), \tag{37}$$

which holds for the duration between the two sending events belonging to FME'$_k$ and FME$_{k+1}$. This duration is included in $[t^k, t^{k+1}]$, so we can apply the bounds of (36) and (35) upon them of (37). An extraction of Δt_q yields

$$\frac{\Delta T_q}{1 + \theta_q^- + \frac{\sigma_q P_{\text{CRA}}}{1-2\rho_q} + \mathcal{O}\left((||\mathbf{R}_q|| + \sigma_q P_{\text{CRA}})^2\right)} \leq \Delta t_q \leq \frac{\Delta T_q}{1 - \theta_q^+ - \frac{\sigma_q P_{\text{CRA}}}{1-2\rho_q} - \mathcal{O}\left((||\mathbf{R}_q|| + \sigma_q P_{\text{CRA}})^2\right)}.$$

Exploiting (20) gives us the intermediate $\Delta t_{p,q}$ bounds. They are subsequently mapped onto clock $\mathcal{C}_p$ in order to obtain the aspired bounds on $\Delta T_{p,q}$. Since $\mathcal{C}_p$ is the object under measurement, we can solely use its possible range of clock rates based on (2) to receive

$$\Delta T_{p,q} \leq \left(\Delta T_q \left(1+\theta_q^+ + \frac{\sigma_q P_{\mathrm{CRA}}}{1-2\rho_q}\right) + \epsilon_{\max}\right)(1+2\rho_p) + \mathcal{O}\left((||\mathbf{R}_q|| + \sigma_q P_{\mathrm{CRA}})^2\right)\Delta T_q \tag{38}$$

and

$$\Delta T_{p,q} \geq \left(\Delta T_q \left(1-\theta_q^- - \frac{\sigma_q P_{\mathrm{CRA}}}{1-2\rho_q}\right) - \epsilon_{\max}\right)(1-2\rho_p) - \mathcal{O}\left((||\mathbf{R}_q|| + \sigma_q P_{\mathrm{CRA}})^2\right)\Delta T_q \tag{39}$$

with the help of Lemma 2 in the permissive way.

After this preparatory work we are ready to attack $\mathbf{Q}_{p,q}$ itself. Let's begin with the length of it given by the formula in Lemma 8, whereby the left and right lengths are equal. Applying (34) and (39) on it yields

$$\begin{aligned}\frac{1}{2}||\mathbf{Q}_{p,q}|| \quad &\leq \quad \frac{(\sigma_p+\sigma_q)P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \frac{\epsilon_{\max}(1+2\rho_p)}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)\left(1-\theta_q^- - \frac{\sigma_q P_{\mathrm{CRA}}}{1-2\rho_q}\right)(1-2\rho_p)} + \\ &\quad \mathcal{O}\left(\frac{(||\mathbf{R}_q||+\sigma_q P_{\mathrm{CRA}})^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + (\sigma_p^2+\sigma_q^2)P_{\mathrm{CRA}}^2 + \sigma_q\epsilon_{\max}\right),\end{aligned}$$

whereby the $\mathcal{O}$-term and $\epsilon_{\max}$-term of (39) was taken away from the second fraction appropriately. As a consequence from Assumption 1, the drift of any non-faulty free running clock $\mathcal{C}_q$ incurred from the oscillator stability σ_q cannot exceed the range of $4\rho_{\max}$ given by (2), thus $\sigma_q P_{\mathrm{CRA}} \leq 4\rho_{\max}$. Also θ_q^- can be safely bounded by $2\rho_{\max}/(1-2\rho_{\max})$ according to Lemma 1. When pushing all $\rho_{\max}$-terms into the denominator, we finally get for the length of the symmetrical quotient rate interval

$$\begin{aligned}\frac{1}{2}||\mathbf{Q}_{p,q}|| \quad &\leq \quad \frac{(\sigma_p+\sigma_q)P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})} + \\ &\quad \mathcal{O}\left(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}\epsilon_{\max}\right),\end{aligned}$$

which proofs item (1) of our Lemma.

In a similar way, we start out with the formula in Lemma 8 for the reference point, apply (34), (38) and (39) on it. This provides an upper bound of

$$\begin{aligned}ref(\mathbf{Q}_{p,q}) \quad &\leq \quad \frac{P_{\mathrm{CRA}}-P_{\mathrm{CSA}}+B}{P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B}\cdot\frac{1}{(1-\theta_q^- - \frac{\sigma_q P_{\mathrm{CRA}}}{1-2\rho_q})(1-2\rho_p)} + \frac{\epsilon_{\max}}{P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B} + \\ &\quad \mathcal{O}\left(\frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \frac{\epsilon_{\max}\rho_{\max}}{P_{\mathrm{CRA}}} + (||\mathbf{R}_q||+\sigma_q P_{\mathrm{CRA}})^2\right)\end{aligned}$$

and a similar lower bound. Noting that $B \ll (P_{\mathrm{CRA}} - P_{\mathrm{CSA}})$ and using the asymptotic approximation $(a \pm x)/(a \mp x) = 1 \pm 2x/a + \mathcal{O}(x^2)$ and $(1 \pm x)^{-1} = 1 \mp x + \mathcal{O}(x^2)$ both valid for $x \to 0$, we can make further simplifications on the above bounds, which results in

$$ref(\mathbf{Q}_{p,q}) \quad \leq \quad 1 + \theta_q^- + 2\rho_p + \frac{\sigma_q P_{\mathrm{CRA}}}{1-2\rho_q} + \frac{2B+\epsilon_{\max}}{P_{\mathrm{CRA}}-P_{\mathrm{CSA}}} +$$

$$\mathcal{O}\left(\frac{(B+\epsilon_{\max})\rho_{\max}}{P_{\mathrm{CRA}}}+\frac{(B+\epsilon_{\max})^2}{P_{\mathrm{CRA}}^2}+(\rho_{\max}+\sigma_{\max}P_{\mathrm{CRA}})^2\right)$$

and

$$\begin{aligned} ref(\mathbf{Q}_{p,q}) \;\geq\; & 1-\theta_q^{+}-2\rho_p-\frac{\sigma_q P_{\mathrm{CRA}}}{1-2\rho_q}-\frac{2B+\epsilon_{\max}}{P_{\mathrm{CRA}}-P_{\mathrm{CSA}}}- \\ & \mathcal{O}\left(\frac{(B+\epsilon_{\max})\rho_{\max}}{P_{\mathrm{CRA}}}+\frac{(B+\epsilon_{\max})^2}{P_{\mathrm{CRA}}^2}+(\rho_{\max}+\sigma_{\max}P_{\mathrm{CRA}})^2\right). \end{aligned}$$

A translation into the interval notation finishes up the proof. □

Lemma 12 (Interval Multiplication Bounds) *If* $\mathbf{I}=[x,1,y]$ *and* $\mathbf{J}=[r\pm u]$ *then*

(1) $align(\mathbf{J})+left(\mathbf{J})align(\mathbf{I})\subseteq align(\mathbf{I}\cdot\mathbf{J})\subseteq align(\mathbf{J})+right(\mathbf{J})align(\mathbf{I})$

(2) $ref(\mathbf{I}\cdot\mathbf{J})=ref(\mathbf{J})$

Proof First we have $align(\mathbf{I}\cdot\mathbf{J})=[u+x(r-u),0,u+y(r+u)]$. The lower bounding interval is given by $[0\pm u]+(r-u)[x,0,y]$ and the upper one by $[0\pm u]+(r+u)[x,0,y]$. A resubstitution according to Definition 3 delivers the claimed result. The second item is a trivial conclusion from the interval multiplication definition. □

Lemma 13 (Rate Interval Dissemination) *If* $\mathbf{R}_q$ *is a correct local rate interval at remote node* q *at* t^k *and* $\mathbf{Q}_{p,q}$ *is the involved quotient rate interval, then the remote rate interval* $\mathbf{R}_{p,q}$ *at local node* p *has the following properties at* t^{k+1}*:*

(1) $align(\mathbf{R}_{p,q})\subseteq\left(1+2\rho_p+\frac{(3\sigma_q+\sigma_p)P_{\mathrm{CRA}}}{2(1-2\rho_q)}+\frac{2(B+\epsilon_{\max})}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)align(\mathbf{R}_q)+$
$\left[0\pm\left(||\mathbf{R}_q||^2+\frac{\sigma_p P_{\mathrm{CRA}}}{1-2(\rho_p+\rho_q)}+\frac{\sigma_q P_{\mathrm{CRA}}(1+\rho_p+\rho_q)}{1-4\rho_q}+\frac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\right]+$
$\left[0\pm\mathcal{O}\left(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}}+\frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2}+\sigma_{\max}^2P_{\mathrm{CRA}}^2+\sigma_{\max}(B+\Pi_{\max}+\epsilon_{\max})\right)\right]$ *for* $p\neq q$*, and*
$align(\mathbf{R}_{p,p})\subseteq align(\mathbf{R}_p)+\left[0\pm\frac{\sigma_p P_{\mathrm{CRA}}}{1-2\rho_p}\right]+\left[0\pm\mathcal{O}\left(\sigma_{\max}(||\mathbf{R}_p||P_{\mathrm{CRA}}+\Pi_{\max})+\sigma_p^2P_{\mathrm{CRA}}^2\right)\right]$

(2) $ref(\mathbf{R}_{p,q})\in swap(\mathbf{R}_q)+\left[0\pm\left(2\rho_p+\frac{\sigma_q P_{\mathrm{CRA}}}{1-2\rho_q}+\frac{2B+\epsilon_{\max}}{P_{\mathrm{CRA}}-P_{\mathrm{CSA}}}\right)\right]+$
$\left[0\pm\mathcal{O}\left(\frac{(B+\epsilon_{\max})\rho_{\max}}{P_{\mathrm{CRA}}}+\frac{(B+\epsilon_{\max})^2}{P_{\mathrm{CRA}}^2}+(\rho_{\max}+\sigma_{\max}P_{\mathrm{CRA}})^2\right)\right]$ *for* $p\neq q$*, and*
$ref(\mathbf{R}_{p,p})=1$.

Proof Item (1) targets the alignment of the remote rate interval $\mathbf{R}_{p,q}$, whose computation is given at action #6 in Algorithm 1. The alignment of multiplicand $align(\mathbf{R}_q+[0\pm\sigma_q(D'+L)/(1-2\rho_q)])$ can be easily expressed as $[\theta_q^-+\vartheta_q,0,\theta_q^++\vartheta_q]$ with

$$\begin{aligned} \vartheta_q &= \frac{\sigma_q(P_{\mathrm{CSA}}-(1+2\rho_{\max})\eta_{\max})}{1-2\rho_q} \qquad (40) \\ &\leq \frac{\sigma_q P_{\mathrm{CSA}}}{1-2\rho_q}, \end{aligned}$$

since $D' + L = P_{\mathrm{CSA}} - D + L = P_{\mathrm{CSA}} - (1 + 2\rho_{\max})\eta_{\max}$ from (28), (29) and Lemma 10. The properties of multiplier $\mathbf{Q}_{p,q}$ are given by Lemma 11, so we just need to string them together by virtue of Lemma 12. The remaining component to prepare is the right egde of $\mathbf{Q}_{p,q}$, which is

$$right(\mathbf{Q}_{p,q}) = 1+\theta_q^- +\vartheta_{p,q}+\mathcal{O}\left(\frac{(B+\epsilon_{\max})\rho_{\max}}{P_{\mathrm{CRA}}} + \frac{(B+\epsilon_{\max})^2}{P_{\mathrm{CRA}}^2} + (\rho_{\max} + \sigma_{\max}P_{\mathrm{CRA}})^2 + \sigma_{\max}\epsilon_{\max}\right)$$

with

$$\begin{aligned}
\vartheta_{p,q} &= 2\rho_p + \frac{\sigma_q P_{\mathrm{CRA}}}{1-2\rho_q} + \frac{2B+\epsilon_{\max}}{P_{\mathrm{CRA}} - P_{\mathrm{CSA}}} + \frac{(\sigma_p+\sigma_q)P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}} - P_{\mathrm{CSA}} - B)(1-10\rho_{\max})} \\
&\leq 2\rho_p + \frac{(3\sigma_q+\sigma_p)P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \frac{2(B+\epsilon_{\max})}{(P_{\mathrm{CRA}} - P_{\mathrm{CSA}} - B)(1-10\rho_{\max})}.
\end{aligned}$$

Now we are ready to attack the upper bound on the alignment of $\mathbf{R}_{p,q}$. According to Lemma 12 and recalling #6 in Algorithm 1, we have

$$align(\mathbf{R}_{p,q}) \subseteq align(\mathbf{Q}_{p,q})+right(\mathbf{Q}_{p,q})\ align\left(\mathbf{R}_q + \left[0 \pm \frac{\sigma_q(D'+L)}{1-2\rho_q}\right]\right)+\left[0 \pm ref(\mathbf{Q}_{p,q})\frac{\sigma_p D}{1-2\rho_p}\right],$$

which can be casted into

$$\begin{aligned}
align(\mathbf{R}_{p,q}) \ \subseteq\ & (1+\theta_q^- + \vartheta_{p,q})\left[\theta_q^- + \vartheta_q, 0, \theta_q^+ + \vartheta_q\right] + \\
& \left[0 \pm \left(\frac{(\sigma_p+\sigma_q)P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}} - P_{\mathrm{CSA}} - B)(1-10\rho_{\max})} + \frac{\sigma_p D}{1-2\rho_p}\right)\right] + \\
& \left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(B+\epsilon_{\max})\right)\right]
\end{aligned} \tag{41}$$

Next we focus on the first interval of (41). Since $0 \leq \theta_q^- \leq 2\rho_q/(1-2\rho_q)$ from Lemma 1, we can interfere that

$$(1+\theta_q^- + \vartheta_{p,q})(\theta_q^\pm + \vartheta_q) \leq (1+\vartheta_{p,q})\theta_q^\pm + (\theta_q^- + \theta_q^+)^2 + (1+\frac{2\rho_q}{1-2\rho_q} + \vartheta_{p,q})\vartheta_q,$$

where the last term can be placed to the deterioration interval of (41). By reshaping this interval, we get for its left/right length

$$\begin{aligned}
l_{p,q} &= \frac{(\sigma_p+\sigma_q)P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}} - P_{\mathrm{CSA}} - B)(1-10\rho_{\max})} + \frac{\sigma_p D}{1-2\rho_p} + (1+\frac{2\rho_q}{1-2\rho_q} + \vartheta_{p,q})\vartheta_q \\
&\leq \sigma_p\left(\frac{P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \frac{D}{1-2\rho_p}\right) + \sigma_q\left(\frac{P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \frac{(1+2\rho_p+2\rho_q)P_{\mathrm{CSA}}}{1-4\rho_q}\right) + \\
&\quad \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}} - P_{\mathrm{CSA}} - B)(1-10\rho_{\max})} + \mathcal{O}\left(\sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(B+\epsilon_{\max})\right) \\
&\leq \frac{\sigma_p P_{\mathrm{CRA}}}{1-2(\rho_p+\rho_q)} + \frac{\sigma_q P_{\mathrm{CRA}}(1+\rho_p+\rho_q)}{1-4\rho_q} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}} - P_{\mathrm{CSA}} - B)(1-10\rho_{\max})} + \\
&\quad \mathcal{O}\left(\sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(B+\epsilon_{\max})\right),
\end{aligned}$$

since $P_{\mathrm{CRA}} \geq 2P_{\mathrm{CSA}} \geq 2D$ from (28) and below. After all, we can restate (41) as

$$\begin{aligned} align(\mathbf{R}_{p,q}) \quad \subseteq \quad & (1+\vartheta_{p,q})\, align(\mathbf{R}_q) + \left[0 \pm (||\mathbf{R}_q||^2 + l_{p,q})\right] + \\ & \left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2 \epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(B+\epsilon_{\max})\right)\right]. \end{aligned}$$

Plugging in the bounds for our temporary variables $\vartheta_{p,q}$ and $l_{p,q}$, and adding $\mathcal{O}(\sigma_{\max}\Pi_{\max})$ to make it correct at t^{k+1} according to Definition 13 finishes item (1) for case $q \neq p$. The other case is a trivial consequence of Lemma 3 and Definition 13.

Item (2) follows immediately from Lemma 12, since the deterioration terms don't affect the reference points. Therefore it is sufficient to take over the interval from item (2) of Lemma 11 for case $q \neq p$. The other case is also trivial. □

Lemma 14 (Consonance Interval Dissemination) *Let $\mathcal{N}^k$ be the set of non-faulty nodes during round k. If the set of correct local rate intervals $\boldsymbol{\mathcal{R}}_0 = \{\mathbf{R}_p | p \in \mathcal{N}^k\}$ is $\boldsymbol{\gamma}_0$-correct w.r.t. internal global rate $\omega(t^k)$ at t^k and $\mathbf{Q}_{p,q}$ is the involved quotient rate interval, then the remote rate interval $\mathbf{R}_{p,q}$ for $p, q \in \mathcal{N}^k$ have the following properties at t^{k+1}:*

(1) Any remote rate interval $\mathbf{R}_{p,q}$ is $\boldsymbol{\gamma}_{p,q}$-correct with

$$\begin{aligned} &\boldsymbol{\gamma}_{p,q} \subseteq \left(1 + 2\rho_p + \tfrac{(3\sigma_q+\sigma_p)P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \tfrac{2(B+\epsilon_{\max})}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\boldsymbol{\gamma}_0 + \\ &\left[0 \pm \left(||\boldsymbol{\gamma}_0||^2 + \tfrac{\sigma_p P_{\mathrm{CRA}}}{1-2(\rho_p+\rho_q)} + \tfrac{\sigma_q P_{\mathrm{CRA}}(1+\rho_p+\rho_q)}{1-4\rho_q} + \tfrac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\right] + \\ &\left[0 \pm \mathcal{O}\left(\tfrac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \tfrac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(B + \Pi_{\max} + \epsilon_{\max})\right)\right] \textit{ for } p \neq q, \textit{ and} \\ &\boldsymbol{\gamma}_{p,p} \subseteq \boldsymbol{\gamma}_0 + \left[0 \pm \tfrac{\sigma_p P_{\mathrm{CRA}}}{1-2\rho_p}\right] + \left[0 \pm \mathcal{O}\left(\sigma_{\max}(||\boldsymbol{\gamma}_0||P_{\mathrm{CRA}} + \Pi_{\max}) + \sigma_p^2 P_{\mathrm{CRA}}^2\right)\right] \end{aligned}$$

(2) The set of remote rate intervals $\boldsymbol{\mathcal{R}}_p = \{\mathbf{R}_{p,q} | q \in \mathcal{N}^k\}$ at a receiving node p supplied by broadcasting nodes $q \in \mathcal{N}^k$ is $\boldsymbol{\gamma}_p^H$-correct with

$$\begin{aligned} &\boldsymbol{\gamma}_p^H \subseteq \left(1 + 2\rho_p + \tfrac{(3\sigma_{\max}+\sigma_p)P_{\mathrm{CRA}}}{2(1-2\rho_{\max})} + \tfrac{2(B+\epsilon_{\max})}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\boldsymbol{\gamma}_0 + \\ &\left[0 \pm \left(||\boldsymbol{\gamma}_0||^2 + \tfrac{\sigma_p P_{\mathrm{CRA}}}{1-2(\rho_p+\rho_{\max})} + \tfrac{\sigma_{\max} P_{\mathrm{CRA}}(1+\rho_p+\rho_{\max})}{1-4\rho_{\max}} + \tfrac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\right] + \\ &\left[0 \pm \mathcal{O}\left(\tfrac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \tfrac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(||\boldsymbol{\gamma}_0||P_{\mathrm{CRA}} + B + \Pi_{\max} + \epsilon_{\max})\right)\right]. \end{aligned}$$

(3) The set of all remote rate intervals $\boldsymbol{\mathcal{R}} = \bigcup_{p \in \mathcal{N}^k} \mathcal{R}_p$ at non-faulty nodes is $\boldsymbol{\gamma}_H$-correct with

$$\begin{aligned} &\boldsymbol{\gamma}_H \subseteq \left(1 + 2\rho_{\max} + \tfrac{2\sigma_{\max}P_{\mathrm{CRA}}}{1-2\rho_{\max}} + \tfrac{2(B+\epsilon_{\max})}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\boldsymbol{\gamma}_0 + \\ &\left[0 \pm \left(||\boldsymbol{\gamma}_0||^2 + \tfrac{2\sigma_{\max} P_{\mathrm{CRA}}(1+\rho_{\max})}{1-4\rho_{\max}} + \tfrac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\right] + \\ &\left[0 \pm \mathcal{O}\left(\tfrac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \tfrac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(||\boldsymbol{\gamma}_0||P_{\mathrm{CRA}} + B + \Pi_{\max} + \epsilon_{\max})\right)\right]. \end{aligned}$$

(4) The set of remote rate intervals $\boldsymbol{\mathcal{R}}^q = \{\mathbf{R}_{p,q} | p \in \mathcal{N}^k\}$ at receiving nodes $p \in \mathcal{N}^k$ from a broadcasting node q is $\boldsymbol{\gamma}^q$-correct with

$$\begin{aligned} &\boldsymbol{\gamma}^q \subseteq \left(1 + 2\rho_{\max} + \tfrac{(3\sigma_q+\sigma_{\max})P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \tfrac{2(B+\epsilon_{\max})}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\boldsymbol{\gamma}_0 + \\ &\left[0 \pm \left(||\boldsymbol{\gamma}_0||^2 + \tfrac{\sigma_{\max} P_{\mathrm{CRA}}}{1-2(\rho_{\max}+\rho_q)} + \tfrac{\sigma_q P_{\mathrm{CRA}}(1+\rho_{\max}+\rho_q)}{1-4\rho_q} + \tfrac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\right] + \end{aligned}$$

$$\left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(||\boldsymbol{\gamma}_0|| P_{\mathrm{CRA}} + B + \Pi_{\max} + \epsilon_{\max})\right)\right]$$

and $\boldsymbol{\gamma}_I$*-consonant with*

$$\boldsymbol{\gamma}_I \subseteq \left[0 \pm \left(\frac{\sigma_{\max} P_{\mathrm{CRA}}}{1-2(\rho_{\max}+\rho_q)} + \frac{\sigma_q P_{\mathrm{CRA}}(1+\rho_{\max}+\rho_q)}{1-4\rho_q} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\right] +$$
$$\left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(B + \Pi_{\max} + \epsilon_{\max})\right)\right]$$

Proof Item (1) is analogous to item (1) of Lemma 13, but instead of a rate intervals, the associated consonance intervals are analyzed. Suppose node $q \in \mathcal{N}^k$ maintains a correct local rate interval $\mathbf{R}_q$ at t^k, which is also $\boldsymbol{\gamma}_0$-correct expressing that $\omega^k \in v_p(t^k)(1+\boldsymbol{\gamma}_0)$, cf. Definition 6. The local rate interval $\mathbf{R}_q + [0 \pm \sigma_q(D'+L)/(1-2\rho_q)]$ is $(\boldsymbol{\gamma}_0 + [0 \pm \sigma_q P_{\mathrm{CSA}}/(1-2\rho_q)])$-correct, see (41) and Lemma 6, when the message of FME'$_k$ leaves node q. Let $p \in \mathcal{N}^k$ be the peer node with $p \neq q$ computing remote rate interval $\mathbf{R}_{p,q}$ for t^{k+1} at action #6. It remains to quantify the $\boldsymbol{\gamma}_{p,q}$-correctness of $\mathbf{R}_{p,q}$. Due to Corollary 2, we can use the same line of reasoning as for $align(\mathbf{R}_{p,q})$ in Lemma 13 in order to find a bounding interval of $\boldsymbol{\gamma}_{p,q}$, since

$$\boldsymbol{\gamma}_{p,q} \subseteq \left(\boldsymbol{\gamma}_0 + \left[0 \pm \frac{\sigma_q P_{\mathrm{CSA}}}{1-2\rho_q}\right]\right) \cdot \mathbf{Q}_{p,q} + \left[0 \pm ref(\mathbf{Q}_{p,q})\frac{\sigma_p D}{1-2\rho_p}\right].$$

Eventually, we arrive at $\omega^k \in v_q(t^{k+1})(ref(\mathbf{R}_{p,q}) + \boldsymbol{\gamma}_{p,q})$ with the properties from item (2) of Lemma 13 on $ref(\mathbf{R}_{p,q})$ and from item (1) on $\boldsymbol{\gamma}_{p,q}$ after applying two substitutions, viz. $align(\mathbf{R}_{p,q})$ becomes to $\boldsymbol{\gamma}_{p,q}$ and $align(\mathbf{R}_q)$ to $\boldsymbol{\gamma}_0$. Lastely, case $p = q$ is a trival consequence of Lemma 6 and Definiton 13.

Since each remote rate interval is $\boldsymbol{\gamma}_{p,q}$-correct as asserted in item (1), a straight majorization over the broadcasting nodes provides the claimed $\boldsymbol{\gamma}_p^H$-correctness of set $\boldsymbol{\mathcal{R}}_p$. More specifically, calculating $\boldsymbol{\gamma}_p^H = \bigcup_{q\in\mathcal{N}^k\setminus\{p\}} \boldsymbol{\gamma}_{p,q}$ results in the formula as given in item (2), whereas the consonance interval $\boldsymbol{\gamma}_{p,p}$ is also subsumed, since $\boldsymbol{\gamma}_{p,p} \subseteq \boldsymbol{\gamma}_p^H$ holds.

Item (3) embraces all appearing remote rate intervals at non-faulty nodes, whose $\boldsymbol{\gamma}^H$-correctness w.r.t. ω^k can easily be proved by doing the union $\bigcup_{p\in\mathcal{N}^k} \boldsymbol{\gamma}_p^H$. Effortlessly, we arrive at the desired formula.

Again, each remote rate interval is $\boldsymbol{\gamma}_{p,q}$-correct as given in item (1), so a majorization over the receiving nodes, formally $\boldsymbol{\gamma}^q = \bigcup_{p\in\mathcal{N}^k\setminus\{q\}} \boldsymbol{\gamma}_{p,q}$, results in the formula given in the first part of item (4), whereas $\boldsymbol{\gamma}_{q,q} \subseteq \boldsymbol{\gamma}^q$ holds as well.

For the second part, we have to calculate the consonance of the set of remote rate intervals $\boldsymbol{\mathcal{R}}^q$ stemming from a common broadcasting node q. For that purpose, we attach a pseudo consonance interval $\boldsymbol{\kappa}_q$ to local rate interval $\mathbf{R}_q$. It is initialized to $\emptyset$ when the first message leaves node q during FME'$_k$. Covering a point-to-point and a broadcast-type network uniformly, we deteriorate it by the maximum logical braodcast delay B to take care for the last leaving message during FME'$_k$, hence we end up with

$$\boldsymbol{\kappa}_q = \left[0 \pm \frac{\sigma_q B}{1-2\rho_q}\right] + \left[0 \pm \mathcal{O}\left(\sigma_q^2 B^2\right)\right].$$

Replacing $\mathbf{R}_q$ by $\boldsymbol{\kappa}_q$ in item (1) of Lemma 13, yields the corresponding pseudo consonance interval $\boldsymbol{\kappa}_{p,q}$. Carrying out all simplification steps, in particular the $\mathcal{O}$-operations, we obtain

$$\boldsymbol{\kappa}_{p,q} \subseteq \left[0 \pm \left(\frac{\sigma_p P_{\mathrm{CRA}}}{1-2(\rho_p+\rho_q)} + \frac{\sigma_q P_{\mathrm{CRA}}(1+\rho_p+\rho_q)}{1-4\rho_q} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\right] +$$

$$\left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2 \epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(B + \Pi_{\max} + \epsilon_{\max})\right)\right],$$

for $p \neq q$, and

$$\boldsymbol{\kappa}_{q,q} \subseteq \left[0 \pm \frac{\sigma_q P_{\mathrm{CRA}}}{1 - 2\rho_q}\right] + \left[0 \pm \mathcal{O}\left(\sigma_{\max}(B + \Pi_{\max}) + \sigma_{\max}^2 P_{\mathrm{CRA}}^2\right)\right].$$

A majorization over the receiving nodes provides the $\boldsymbol{\gamma}_I$-consonance of $\boldsymbol{\mathcal{R}}^q$, hence $\boldsymbol{\gamma}_I = \bigcup_{p \in \mathcal{N}^k \setminus \{q\}} \boldsymbol{\kappa}_{p,q}$ leads to the formula given in the second part of item (4), noting that $\boldsymbol{\kappa}_{q,q} \subseteq \boldsymbol{\gamma}_I$.

This eventually completes the proof of Lemma 14. □

Note that we have managed to express the properties of the remote rate interval $\mathbf{R}_{p,q}$ at t^{k+1} basically linear in the terms of the the local rate interval $\mathbf{R}_q$ and consonance interval $\boldsymbol{\gamma}_0$ at t^k. In case of ideal conditions, i.e. stable clocks and zero uncertainties of the communication subsystem, we can roughly say that:

- $align(\mathbf{R}_{p,q}) \subseteq (1 + 2\rho_p) align(\mathbf{R}_q)$ for $q \neq p$, and $align(\mathbf{R}_{p,p}) = align(\mathbf{R}_p)$
- $ref(\mathbf{R}_{p,q}) \in swap(\mathbf{R}_q) + [0 \pm 2\rho_p]$ for $q \neq p$, and $ref(\mathbf{R}_{p,p}) = 1$
- $\mathbf{R}_{p,q}$ is $\boldsymbol{\gamma}_{p,q}$-correct with $\boldsymbol{\gamma}_{p,q} \subseteq (1 + 2\rho_p)\boldsymbol{\gamma}_0$ for $p \neq q$, and $\boldsymbol{\gamma}_{p,p} = \boldsymbol{\gamma}_0$
- $\boldsymbol{\mathcal{R}}_p$ is $\boldsymbol{\gamma}_p^H$-correct with $\boldsymbol{\gamma}_p^H \subseteq (1 + 2\rho_p)\boldsymbol{\gamma}_0$
- $\boldsymbol{\mathcal{R}}$ is $\boldsymbol{\gamma}_H$-correct with $\boldsymbol{\gamma}_H \subseteq (1 + 2\rho_{\max})\boldsymbol{\gamma}_0$
- $\boldsymbol{\mathcal{R}}^q$ is $\boldsymbol{\gamma}^q$-correct with $\boldsymbol{\gamma}^q \subseteq (1 + 2\rho_{\max})\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_I$-consonant with $\boldsymbol{\gamma}_I = \emptyset$

This finishes the analysis of disseminating rate and consonance intervals, which have their origins as local rate intervals and are avaiable as remote rate intervals at the end of a particular round. Subsequently they are fed into the validation and convergence function, producing local rate intervals for the next round after adjusting the clock rate accordingly. The remaining sections focus on the evaluation of theses functions based on the input intervals characterized by Lemma 13 and 14.

5.2 Rate Adjustments

We investigate the amounts of rate adjustments administered at the end of each round by changing instanteneously the oscillator-clock coupling factor S_p in action #9 of Algorithm CRA. Since this change happens in a multiplicative way, we aim to find bounds on the ratio between the new and old one. An additional sanity check for feasible values of S_p helps to exclude faulty clocks. Our analysis is grounded on the consonance preservation function $\mathcal{CP}(\cdot)$ of the utilized convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ in algorithm CRA, cf. Definition 11.

Lemma 15 (Rate Adjustments) *Let $\mathcal{N}^k$ be the set of non-faulty nodes during round k. If the set of correct local rate intervals $\boldsymbol{\mathcal{R}}_0 = \{\mathbf{R}_p | p \in \mathcal{N}^k\}$ is $\boldsymbol{\gamma}_0$-correct w.r.t. internal global rate $\omega(t^k)$ at t^k, then the rate adjustment at t^{k+1}, expressed as the ratio between the new and old oscillator-clock coupling factor, is limited by*

$$\begin{aligned} \frac{S_p^{k+1}}{S_p^k} \in\ & 1 + \boldsymbol{\gamma}_0 + \left[0 \pm \frac{\sigma_p P_{\mathrm{CRA}}}{1 - 2\rho_p}\right] + swap\left(\mathcal{CP}(\boldsymbol{\gamma}_{p,1} \ldots, \boldsymbol{\gamma}_{p,n}; \boldsymbol{\gamma}_H; \boldsymbol{\gamma}_I; \ldots)\right) + \\ & \left[0 \pm \mathcal{O}\left(\sigma_{\max}(||\boldsymbol{\gamma}_0|| P_{\mathrm{CRA}} + \Pi_{\max}) + \sigma_p^2 P_{\mathrm{CRA}}^2\right)\right] \end{aligned} \tag{42}$$

for node $p \in \mathcal{N}^k$, whereby $\mathcal{CP}(\boldsymbol{\gamma}_{p,1} \dots, \boldsymbol{\gamma}_{p,n}; \boldsymbol{\gamma}_H; \boldsymbol{\gamma}_I; \dots)$ is a weakly monotonic consonance preservation function of convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ subject to fault model $\mathcal{F}$ supplied with

- $\boldsymbol{\gamma}_{p,q} = \left(1 + 2\rho_p + \frac{(3\sigma_q+\sigma_p)P_{\mathrm{CRA}}}{2(1-2\rho_q)} + \frac{2(B+\epsilon_{\max})}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\boldsymbol{\gamma}_0 +$
$\left[0 \pm \left(||\boldsymbol{\gamma}_0||^2 + \frac{\sigma_p P_{\mathrm{CRA}}}{1-2(\rho_p+\rho_q)} + \frac{\sigma_q P_{\mathrm{CRA}}(1+\rho_p+\rho_q)}{1-4\rho_q} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\right] +$
$\left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(B + \Pi_{\max} + \epsilon_{\max})\right)\right]$,
- $\boldsymbol{\gamma}_H = \left(1 + 2\rho_{\max} + \frac{2\sigma_{\max}P_{\mathrm{CRA}}}{1-2\rho_{\max}} + \frac{2(B+\epsilon_{\max})}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\boldsymbol{\gamma}_0 +$
$\left[0 \pm \left(||\boldsymbol{\gamma}_0||^2 + \frac{2\sigma_{\max}P_{\mathrm{CRA}}(1+\rho_{\max})}{1-4\rho_{\max}} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\right] +$
$\left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(||\boldsymbol{\gamma}_0||P_{\mathrm{CRA}} + B + \Pi_{\max} + \epsilon_{\max})\right)\right]$, *and*
- $\boldsymbol{\gamma}_I = \left[0 \pm \left(\frac{\sigma_{\max}P_{\mathrm{CRA}}}{1-2(\rho_{\max}+\rho_q)} + \frac{\sigma_q P_{\mathrm{CRA}}(1+\rho_{\max}+\rho_q)}{1-4\rho_q} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}}-P_{\mathrm{CSA}}-B)(1-10\rho_{\max})}\right)\right] +$
$\left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(B + \Pi_{\max} + \epsilon_{\max})\right)\right]$.

Proof At the beginning of round k, suppose clock $\mathcal{C}_p$ of node $p \in \mathcal{N}^k$ is steered with coupling factor S_p^k, and the corresponding local rate interval $\mathbf{R}_p$ is $\boldsymbol{\gamma}_0$-correct w.r.t. ω^k. From item (1) of Lemma 14 we know that $\mathbf{R}_{p,p}$ is $\boldsymbol{\gamma}_{p,p}$-correct as shown there. Hence, if $v_p(t^{k+1})$ is the clock rate just before the rate adjustment takes place to switch over to round $k+1$, it follows that

$$\omega^k \in v_p(t^{k+1})\left(1 + \boldsymbol{\gamma}_0 + \left[0 \pm \frac{\sigma_p P_{\mathrm{CRA}}}{1-2\rho_p}\right] + \left[0 \pm \mathcal{O}\left(\sigma_{\max}(||\boldsymbol{\gamma}_0||P_{\mathrm{CRA}} + \Pi_{\max}) + \sigma_p^2 P_{\mathrm{CRA}}^2\right)\right]\right). \quad (43)$$

Furthermore, node p computes a new correct rate interval $\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}}$ valid for t^{k+1} by virtue of the ideal validation function $\mathcal{VAL}_{\mathcal{F}}(\cdot)$ and the convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ tailored to fault model $\mathcal{F}$. The characterization of $\mathcal{CV}_{\mathcal{F}}(\cdot)$ with the consonance preservation function $\mathcal{CP}(\cdot)$ ensures that $\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}}$ is $\mathcal{CP}(\boldsymbol{\gamma}_{p,1} \dots, \boldsymbol{\gamma}_{p,n}; \boldsymbol{\gamma}_H; \boldsymbol{\gamma}_I; \dots)$-correct w.r.t. ω^k, cf. Definition 11. By virtue of the weak monotonicity of $\mathcal{CP}(\cdot)$, it holds that

$$\omega^k \in v_p(t^{k+1})\left(ref(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}}) + \mathcal{CP}(\boldsymbol{\gamma}_{p,1} \dots, \boldsymbol{\gamma}_{p,n}; \boldsymbol{\gamma}_H; \boldsymbol{\gamma}_I; \dots)\right) \quad (44)$$

with the above calculated intervals for $\boldsymbol{\gamma}_{p,1} \dots, \boldsymbol{\gamma}_{p,n}$, $\boldsymbol{\gamma}_H$ and $\boldsymbol{\gamma}_I$ taken over as bounds from item (1), (3) and (4) of Lemma 14. Combining (43) and (44) yields

$$\begin{aligned} ref(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}}) \in\ & 1 + \boldsymbol{\gamma}_0 + \left[0 \pm \frac{\sigma_p P_{\mathrm{CRA}}}{1-2\rho_p}\right] + swap\left(\mathcal{CP}(\boldsymbol{\gamma}_{p,1} \dots, \boldsymbol{\gamma}_{p,n}; \boldsymbol{\gamma}_H; \boldsymbol{\gamma}_I; \dots)\right) + \quad (45) \\ & \left[0 \pm \mathcal{O}\left(\sigma_{\max}(||\boldsymbol{\gamma}_0||P_{\mathrm{CRA}} + \Pi_{\max}) + \sigma_p^2 P_{\mathrm{CRA}}^2\right)\right]. \end{aligned}$$

Noting that $S_p^{k+1} = S_p^k ref(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}})$ from (30) or see action #9 in algorithm CRA finishes the proof. □

5.3 Consonance

For internal rate synchronization, we are interested how the consonance of our clocks behave. In particular, we study the sustentation of the consonance, which is determinated by the consonance enhancement function $\mathcal{CE}(\cdot)$ upon $\mathcal{CV}_{\mathcal{F}}(\cdot)$, cf. Definition 12. The following Theorem expresses the lengths of the resulting consonance intervals $\boldsymbol{\gamma}_p$, whereby the conversion to consonance γ can be done via Lemma 4 and 5.

Theorem 1 (Consonance) *Complying to Assumptions 1-5, if Algorithm CRA employs an ideal validation function $\mathcal{VAL}_{\mathcal{F}}(\cdot)$ and a translation invariant, weakly monotonic convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ characterized by a weakly monotonic consonance enhancement function $\mathcal{CE}(\cdot)$ subject to a given fault model $\mathcal{F}$, then the local rate interval $\mathbf{R}_p$ of a non-faulty node p is $\boldsymbol{\gamma}_0$-correct at t^k for $k \geq 0$, whereby $\boldsymbol{\gamma}_0$ is a solution of the equation*

$$||\boldsymbol{\gamma}_0|| = \frac{\mathcal{CE}(\boldsymbol{\gamma}^1 \ldots, \boldsymbol{\gamma}^n; \boldsymbol{\gamma}_H; \boldsymbol{\gamma}_I; \ldots)}{1 - ||\boldsymbol{\gamma}_0||} \tag{46}$$

with

- $\boldsymbol{\gamma}^q = \Big(1 + 2\rho_{\max} + \frac{(3\sigma_q + \sigma_{\max})P_{\text{CRA}}}{2(1-2\rho_q)} + \frac{2(B+\epsilon_{\max})}{(P_{\text{CRA}}-P_{\text{CSA}}-B)(1-10\rho_{\max})}\Big)\boldsymbol{\gamma}_0 +$
$\Big[0 \pm \Big(||\boldsymbol{\gamma}_0||^2 + \frac{\sigma_{\max}P_{\text{CRA}}}{1-2(\rho_{\max}+\rho_q)} + \frac{\sigma_q P_{\text{CRA}}(1+\rho_{\max}+\rho_q)}{1-4\rho_q} + \frac{\epsilon_{\max}}{(P_{\text{CRA}}-P_{\text{CSA}}-B)(1-10\rho_{\max})}\Big)\Big] +$
$\Big[0 \pm \mathcal{O}\Big(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\text{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\text{CRA}}^2} + \sigma_{\max}^2 P_{\text{CRA}}^2 + \sigma_{\max}(B + \Pi_{\max} + \epsilon_{\max})\Big)\Big]$,
- $\boldsymbol{\gamma}_H = \Big(1 + 2\rho_{\max} + \frac{2\sigma_{\max}P_{\text{CRA}}}{1-2\rho_{\max}} + \frac{2(B+\epsilon_{\max})}{(P_{\text{CRA}}-P_{\text{CSA}}-B)(1-10\rho_{\max})}\Big)\boldsymbol{\gamma}_0 +$
$\Big[0 \pm \Big(||\boldsymbol{\gamma}_0||^2 + \frac{2\sigma_{\max}P_{\text{CRA}}(1+\rho_{\max})}{1-4\rho_{\max}} + \frac{\epsilon_{\max}}{(P_{\text{CRA}}-P_{\text{CSA}}-B)(1-10\rho_{\max})}\Big)\Big] +$
$\Big[0 \pm \mathcal{O}\Big(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\text{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\text{CRA}}^2} + \sigma_{\max}^2 P_{\text{CRA}}^2 + \sigma_{\max}(||\boldsymbol{\gamma}_0||P_{\text{CRA}} + B + \Pi_{\max} + \epsilon_{\max})\Big)\Big]$, *and*
- $\boldsymbol{\gamma}_I = \Big[0 \pm \Big(\frac{\sigma_{\max}P_{\text{CRA}}}{1-2(\rho_{\max}+\rho_q)} + \frac{\sigma_q P_{\text{CRA}}(1+\rho_{\max}+\rho_q)}{1-4\rho_q} + \frac{\epsilon_{\max}}{(P_{\text{CRA}}-P_{\text{CSA}}-B)(1-10\rho_{\max})}\Big)\Big] +$
$\Big[0 \pm \mathcal{O}\Big(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\text{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\text{CRA}}^2} + \sigma_{\max}^2 P_{\text{CRA}}^2 + \sigma_{\max}(B + \Pi_{\max} + \epsilon_{\max})\Big)\Big]$.

Proof The above result is established by conducting an induction proof on round k. For ease of presentation, assume a set $\mathcal{N}^k$ of non-faulty nodes during round $k \geq 0$. The key part is to show that if the set of local rate intervals $\{\mathbf{R}_p^k | p \in \mathcal{N}^k\}$ is $\boldsymbol{\gamma}_0'$-correct w.r.t. internal global rate $\omega(t^k)$ at t^k for some $\boldsymbol{\gamma}_0' \subseteq \boldsymbol{\gamma}_0$, then $\mathbf{R}_p^{k+1}$ is $\boldsymbol{\gamma}_0$-correct w.r.t. the newly internal global rate $\omega(t^{k+1})$ at t^{k+1}.

Starting backwards, we know that if $\boldsymbol{\gamma}_0$ satisfies (46) it follows from Definition 12 that the set $\{\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}} | p \in \mathcal{N}^k\}$ is $\boldsymbol{\gamma}^{\mathcal{CV}_{\mathcal{F}}}$-consonant at t^{k+1}, whereas $||\boldsymbol{\gamma}^{\mathcal{CV}_{\mathcal{F}}}|| = ||\boldsymbol{\gamma}_0||(1 - ||\boldsymbol{\gamma}_0||)$. The enforcement of the new clock rate $v_p(t^{k+1})ref(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}})$ at action #9 of algorithm CRA affects the consonance intervals as well, since in accordance with Definition 5 it holds that

$$\bigcap_{p \in \mathcal{N}^k} v_p(t^{k+1})ref(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}})\left(1 + \frac{\boldsymbol{\gamma}^{\mathcal{CV}_{\mathcal{F}}}}{ref(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}})}\right) \neq \emptyset.$$

Therefore, the set $\{\mathbf{R}_p^{k+1} | p \in \mathcal{N}^k\}$ is $\boldsymbol{\gamma}^{\mathcal{CV}_{\mathcal{F}}}/ref(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}})$-consonant at t^{k+1}, whereas

$$\frac{||\boldsymbol{\gamma}^{\mathcal{CV}_{\mathcal{F}}}||}{ref(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}})} \leq ||\boldsymbol{\gamma}_0|| \frac{1 - ||\boldsymbol{\gamma}_0||}{1 - ||\boldsymbol{\gamma}_0|| -}.$$

in conju‘nction with 46.

The set of initial local rate intervals $\mathbf{R}_p^0 = [\rho_p/(1+\rho_p), 1, \rho_p/(1-\rho_p)]$ with $S_p^0 = 1/f_p$ is correct

□

This is only at resychr. point

5.4 Drift

For external rate synchronization, we are interested how the clock drifts evolve. Such a result is useful in two ways: In case that the clock rate validation supplies remote rate intervals from primary nodes, the others serve only as validation intervals. For this operation, it is important to have knowledge about the lengths of the encountered remote rate intervals, bearing in mind their correctness. In lack of remote rate intervals from primary nodes, the drift of our ensemble is about to increase. Again, their lengths tell how much the rates deviate from the ideal rate of 1. The following Theorem expresses the lengths of the resulting local rate intervals $\mathbf{R}_p$ in a recursive way, whereby the conversion to clock drift δ_p can be done via Lemma 1.

Theorem 2 (Drift) *Complying to Assumptions 1-5, if algorithm CRA employs an ideal validation function $\mathcal{VAL}_{\mathcal{F}}(\cdot)$ and a translation invariant, weakly monotonic convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ characterized by a weakly monotonic drift preservation function $\mathcal{DP}(\cdot)$ subject to a given fault model $\mathcal{F}$, then the local rate intervals $\mathbf{R}_1, \ldots, \mathbf{R}_n$ have the following properties:*

(1) The local rate interval $\mathbf{R}_p$ of a non-faulty node p is correct at t^k for $k \geq 0$.

(2) The local rate interval $\mathbf{R}_p$ of a non-faulty node p satisfies $align(\mathbf{R}_p) \subseteq \mathbf{V}_p^k$ at t^k for $k \geq 0$, where $\mathbf{V}_p^0 = \left[\frac{\rho_p}{1+\rho_p}, 0, \frac{\rho_p}{1-\rho_p}\right]$ and $\mathbf{V}_p^{k+1} = MIN(r_p^{\mathcal{CV}_{\mathcal{F}}})\mathcal{DP}(\mathbf{V}_{p,1}^k, \ldots, \mathbf{V}_{p,n}^k; \ldots)$ with

$$\mathbf{V}_{p,q}^k = \left(1 + 2\rho_p + \frac{P_{\text{CRA}}(3\sigma_q+\sigma_p)}{2(1-2\rho_q)} + \frac{2(B+\epsilon_{\max})}{(P_{\text{CRA}}-P_{\text{CSA}}-B)(1-10\rho_{\max})}\right)\mathbf{V}_q^k +$$
$$\left[0 \pm \left(||\mathbf{V}_q^{k-1}||^2 + \frac{\sigma_p P_{\text{CRA}}}{1-2(\rho_p+\rho_q)} + \frac{\sigma_q P_{\text{CRA}}(1+\rho_p+\rho_q)}{1-4\rho_q} + \frac{\epsilon_{\max}}{(P_{\text{CRA}}-P_{\text{CSA}}-B)(1-10\rho_{\max})}\right)\right] +$$
$$\left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2\epsilon_{\max}}{P_{\text{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\text{CRA}}^2} + \sigma_{\max}^2 P_{\text{CRA}}^2 + \sigma_{\max}(B + \Pi_{\max} + \epsilon_{\max})\right)\right].$$

Proof The above results are established by conducting an induction proof on round k. For ease of presentation, assume a set $\mathcal{N}^k$ of non-faulty nodes during round $k \geq 0$.

For item (1) an initial local rate interval $\mathbf{R}_q^0 = [\rho_q/(1+\rho_q), 1, \rho_q/(1-\rho_q)]$ with $S_q^0 = 1/f_q$ is correct according to Lemma 1 if $q \in \mathcal{N}^0$. For an arbitrary $k \geq 1$ choose any $p, q \in \mathcal{N}^k$, whose local rate interval $\mathbf{R}_p^k$ resp. $\mathbf{R}_q^k$ is correct at t_k, with coupling factor S_p^k resp. S_q^k. Let's trace through the way towards remote rate interval $\mathbf{R}_{p,q}^k$ at t^{k+1}. By construction of our algorithm and by consulting Assumption 4, we deduct that

$$C_q(t_q^{\text{FME}'_k}) - C_q(t_q^k) \leq D' + (1+2\rho_{\max})(\tau_{\max} + \lambda_{\max}),$$

whereby $t_q^{\text{FME}'_k}$ denotes the real-time of actual message transmission during FME'$_k$. Lemma 10 provides that $(1+2\rho_{\max})(\tau_{\max}+\lambda_{\max}) \leq L$, so we know that $\mathbf{R}_q^k + [0 \pm \sigma_q(D'+L)/(1-2\rho_q)]$ is correct at $t_q^{\text{FME}'_k}$ by virtue of Lemma 3 and ignoring $\mathcal{O}$-terms. Waiting L logical seconds for message reception guarantees the completion of FME'$_k$. Now we are ready to apply Lemma 9 asserting that the remote rate interval $(\mathbf{R}_q^k + [0 \pm \sigma_q(D'+L)/(1-2\rho_q)]) \cdot \mathbf{Q}_{p,q}$ is correct at $t_p^{\text{FME}_{k+1}}$, when the message arrives at node p during FME_{k+1}. An ensuing deterioration of $[0 \pm ref(\mathbf{Q}_{p,q})\sigma_p D/(1-2\rho_p)]$ makes it correct at t^{k+1}, since

$$C_p(t^{k+1}) - C_p(t_p^{\text{FME}_{k+1}}) \leq D$$

again by construction of our algorithm, and by recalling Lemma 10 and Definition 13. Since messages between non-faulty nodes don't arrive later than $(k+1)P_{\text{CRA}} + L$ during FME_{k+1}, there remains a logical duration of E, sufficient to carry out the computation of CRA. By fixing node

p, we are dealing with a set of correct remote rate interval $\{\mathbf{R}_{p,q}^k | q \in \mathcal{N}^k\}$ at t_{k+1}. Although additional faulty remote rate intervals might be involved, the ideal validation function $\mathcal{VAL}_{\mathcal{F}}(\cdot)$ and the convergence function $\mathcal{CV}_{\mathcal{F}}(\cdot)$ tailored to fault model $\mathcal{F}$ are responsible for computing a new correct rate interval $\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}} = [\theta_p^{\mathcal{CV}_{\mathcal{F}},-}, r_p^{\mathcal{CV}_{\mathcal{F}}}, \theta_p^{\mathcal{CV}_{\mathcal{F}},+}]$. Adjusting the rate by setting the coupling-factor $S_p^{k+1} = S_p^k r_p^{\mathcal{CV}_{\mathcal{F}}}$ entails

$$\left(v_p(t_p^{k+1}) r_p^{\mathcal{CV}_{\mathcal{F}}}\right)\left(1 - \frac{\theta^{\mathcal{CV}_{\mathcal{F}},-}}{r_p^{\mathcal{CV}_{\mathcal{F}}}}\right) \leq 1 \leq \left(v_p(t_p^{k+1}) r_p^{\mathcal{CV}_{\mathcal{F}}}\right)\left(1 + \frac{\theta^{\mathcal{CV}_{\mathcal{F}},+}}{r_p^{\mathcal{CV}_{\mathcal{F}}}}\right),$$

which shows that the new local rate interval $\mathbf{R}_p^{k+1} = \mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}} / r_p^{\mathcal{CV}_{\mathcal{F}}}$ is correct for the new clock rate $r_p^{\mathcal{CV}} v_p(t^{k+1})$. Remarking that the nodes within $\mathcal{N}^k$ are interchangeable finishes the correctness proof.

For item (2) the initial case $k = 0$ is trivial, since $\mathbf{V}_q^0 = \mathbf{R}_q^0$ at t^0 for all $q \in \mathcal{N}^0$. For an arbitrary $k \geq 0$, we assume $align(\mathbf{R}_q^k) \subseteq \mathbf{V}_q^k$ at t^k for all $q \in \mathcal{N}^k$. Using item (1) of Lemma 13, the computed remote rate intervals $\mathbf{R}_{p,q}^k$ at t^k for any $p \in \mathcal{N}^k$ have the property

$$\begin{aligned}
align(\mathbf{R}_{p,q}^k) \subseteq & \left(1 + 2\rho_p + \frac{P_{\mathrm{CRA}}(3\sigma_q + \sigma_p)}{2(1-2\rho_q)} + \frac{2(B + \epsilon_{\max})}{(P_{\mathrm{CRA}} - P_{\mathrm{CSA}} - B)(1 - 10\rho_{\max})}\right) \mathbf{V}_q^k + \\
& \left[0 \pm \left(||\mathbf{V}_q^k||^2 + \frac{\sigma_p P_{\mathrm{CRA}}}{1 - 2(\rho_p + \rho_q)} + \frac{\sigma_q P_{\mathrm{CRA}}(1 + \rho_p + \rho_q)}{1 - 4\rho_q} + \frac{\epsilon_{\max}}{(P_{\mathrm{CRA}} - P_{\mathrm{CSA}} - B)(1 - 10\rho_{\max})}\right)\right] + \\
& \left[0 \pm \mathcal{O}\left(\frac{\rho_{\max}^2 \epsilon_{\max}}{P_{\mathrm{CRA}}} + \frac{\epsilon_{\max}^2}{P_{\mathrm{CRA}}^2} + \sigma_{\max}^2 P_{\mathrm{CRA}}^2 + \sigma_{\max}(B + \Pi_{\max} + \epsilon_{\max})\right)\right],
\end{aligned}$$

which can be abbreviated by $align(\mathbf{R}_{p,q}^k) \subseteq \mathbf{V}_{p,q}^k$. According to Definition 10 concerning the drift preservation, we can derive that

$$align(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}}) \subseteq \mathcal{DP}(\mathbf{V}_{p,1}^k, \ldots, \mathbf{V}_{p,n}^k; \ldots).$$

Since $align(\mathbf{R}_p^{\mathcal{CV}_{\mathcal{F}}}) / r_p^{\mathcal{CV}_{\mathcal{F}}}$ is an including interval of $align(\mathbf{R}_p^{k+1})$, we are in need to find a lower bound on $r_p^{\mathcal{CV}_{\mathcal{F}}}$. This touches the problem of setting the reference point, which is directly tied with internal rate synchronization. □

*** $\mathbf{R}_{p,p}$ nicht vergessen

5.5 Splicing Clock Rate and State Synchronization

Let us review the principles of a clock rate and state synchronization algorithm:

At state resynchronization points, the algorithm achieves a worst case precision $\Pi_{0,\max}$, struggling against message delivery uncertainties and clock granularities. Between these points, the clocks are running free and drift apart by $\boldsymbol{\rho}_{\max} P_{\mathrm{CSA}}$. The rate resynchronization algorithm is in charge of reducing this term, battling against the clock stabilites. There is a tradeoff involved expressed in the ratio $P_{\mathrm{CRA}}/P_{\mathrm{CSA}}$.

Looking at the mutual dependencies between a rate and state synchronization algorithm: We see, that a CSA needs a good rate synchronization, but the CRA only little or even no state synchronization. This is also the key to have a separated analysis, putting a CSA on top of a CRA. Note that there is no conflict with the optimal result of [ST87], since the hardware drift is now affected by the CRA.
** Nicht vergessen die state correction ausweisen, um die unkorrigiert Clock zu bekommen

Let us briefly compare these algorithms:

S state is gut zugaenglich, bei R ist rate nicht (measurement), R hat grosse Zeitkonstanten, S hat kleine (eng, B, gr), R ist mult, inst corr, S is add.and amort corr, R kann Fehler leichter detektieren, S nicht

Usage of CRA:

R kann S helfen auch bei trans delay measurement, faults, initial, R as its own right.

R Verbessern durch pipeling, keine 2 extra FME's,

Conclusion

We crafted an algorithm CRA for synchronizing both internally and externally the rates of clocks in a fault-tolerant distributed system. The not directly observable clock rates are captured by dynamically maintained rate intervals, which are capable to account for the oscillator stability. The algorithm is analog to a round-based one for clock state synchronization, consisting of a method for rate measurement, a validation function to inject external rate references, and a convergence function to compute proper rate corrections. Both functions have fault-tolerance properties tailored towards a realistic system model comprising clocks, processors, and communication networks.

Clock rate synchronization is useful for many distributed applications and can support algorithms for clock state synchronization. Moreover, it has the advantage of an inexpensive implementation, besides the need of a rate adjustable local clock. In the future, we are building a prototype to demonstrate the behavior of the algorithm and to verify the theoretical results. Finally, much work remains to be done to devise and analyze suitable interval-based validation and convergence functions for a specific fault model. Of additional interest is to study the tradeoff between internal and external rate synchronization.

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Glossary

Name	Meaning	Definition	Page
α_p	accuracy of clock $\mathcal{C}_p$	Def. 1	4
B	maximal logical broadcast delay	Lem. 11	28
$C_p(t)$	function of clock $\mathcal{C}_p$	Sec. 2	4
$\mathcal{CE}(\cdot)$	consonance enancement function	Def. 12	24
$\mathcal{CP}(\cdot)$	consonance preservation function	Def. 11	24
$\mathcal{CV}_{\mathcal{F}}(\cdot)$	convergence function referring to fault model $\mathcal{F}$	Sec. 4.2	21
D, D'	delays in algorithm CRA	Sec. 4.1	19
δ_p	drift of clock $\mathcal{C}_p$	Def. 2	4
$\mathcal{DP}(\cdot)$	drift preservation function	Def. 10	21
$\epsilon^{\pm}_{p,q}$	delivery uncertainties	Ass. 5	7
$\epsilon_{\max}$	maximum delivery uncertainty	Ass. 5	7
E	logical duration of computation	Lem. 10	20
$\eta_{\max}$	maximal computation time	Ass. 3	6
$\mathcal{F}$	abstract fault model	Sec. 4.4	24
f_p	nominal frequency of oscillator $\mathcal{O}_p$	Sec. 2.2	4
$f_p(t)$	instentaneous frequency of oscillator $\mathcal{O}_p$	Sec. 2.2	4
γ	consonance of an ensemble of clocks	Def. 2	4
$\boldsymbol{\gamma}$	consonance interval	Def. 5	11
$\mathbf{I}$	asymmetric interval	Def. 3	8
$\boldsymbol{\mathcal{I}}$	ordered set of asymmetric interval	Sec. 3.1	8
$align(\mathbf{I})$	interval alignment	Def. 3	8
$ref(\mathbf{I})$	reference point of interval	Def. 3	8
L	logical duration of an FME	Lem. 10	20
$\lambda_{\max}$	maximum broadcast latency	Ass. 4	7
n	number of nodes	Sec. 4	19
$\omega(t)$	internal global rate	Sec. 3.3	12
P_{CRA}	rate synchronization period	Sec. 4.1	19
P_{CSA}	state synchronization period	Sec. 4.1	19
π	precision of an ensemble of clocks	Def. 1	4
$\Pi_{\max}$	global precision	Ass. 2	6
$\mathbf{Q}_{p,q}$	quotient rate interval	Def. 7	14
$\mathbf{R}_p$	(local) rate interval for clock $\mathcal{C}_p$	Def. 4	9
$\mathbf{R}_{p,q}$	remote rate interval	Lem. 9	17
$\boldsymbol{\mathcal{R}}$	set of (local) rate intervals	Sec. 3.1	8

Name	**Meaning**	**Definition**	**Page**
ρ_p	maximum oscillator drift	Ass. 1	6
$\rho_{\max}$	uniform maximum oscillator drift	Sec. 2.2	4
S_p	coupling factor between $\mathcal{C}_p$ and $\mathcal{O}_p$	Sec. 2.2	4
σ_p	oscillator stability	Ass. 1	6
$\sigma_{\max}$	uniform oscillator stability	Sec. 2.2	4
t	real-time in a Newtonian frame	Sec. 2	4
t^k	begin of round k	Def. 13	27
Δt	real-time duration	Sec. 3.2	9
$\Delta t_{p,q}$	deterministic delivery	Ass. 5	7
$\Delta t_{\max}$	maximum deterministic delivery	Ass. 5	7
T	clock state	Sec. 2	4
ΔT	logical-time duration	Sec. 3.2	9
$\mathbf{T}$	non-empty real-time period	Sec. 2.1	4
$\tau_{\max}$	maximum broadcast operation delay	Ass. 4	7
$\theta_p^{\pm}$	rate drifts for clock $\mathcal{C}_p$	Sec. 3.2	8
U_p, U_q	accumulated state corrections	Alg. 1	22
$v_p(t)$	instentaneous clock rate	Sec. 2.1	4
$\mathcal{VAL}_{\mathcal{F}}(\cdot)$	validation function referring to fault model $\mathcal{F}$	Def. 8	21