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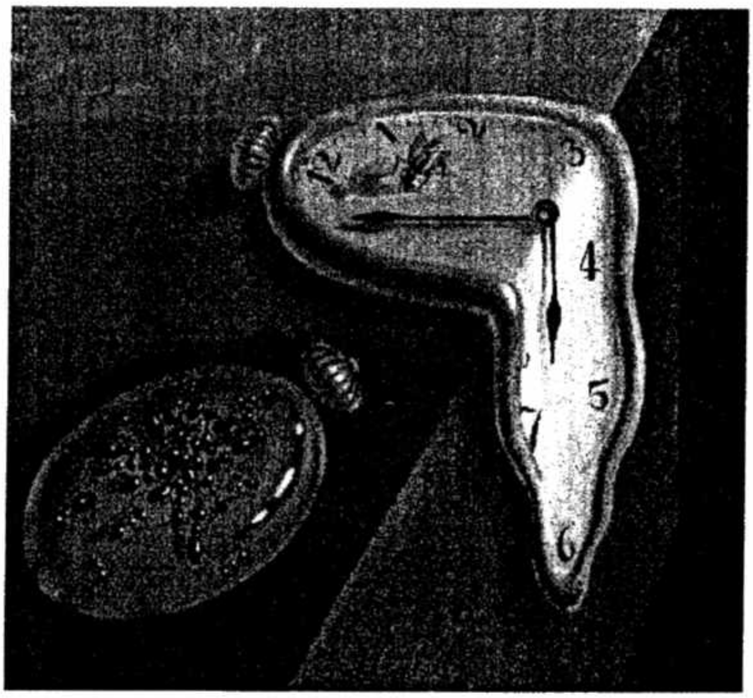

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On the Asymptotics of the average CRI-Length of the Slotted ALOHA Collision Resolution Algorithm[1]

ULRICH SCHMID[2]

Abstract. We provide uniform asymptotic expansions of a finite sum $L_{n,p}$ of essentially geometric type, which arises in the investigation of the average CRI-length of the well-known slotted ALOHA collision resolution algorithm with retransmission probability p. In particular, our investigations establish large regions of uniform validity w.r.t. p as $n \to \infty$. By means of a direct asymptotic method based on the analysis of a certain sum involving binomial coefficients, we obtain an asymptotic expansion of $L_{n,p}$, which is uniformly valid for $p \leq n^{-0.51}/2e$, $n \to \infty$. The application of some well-established methods relying on complex analysis yields another result, this time uniformly valid for $p \geq n^{-0.99}$ as $n \to \infty$.

1. Introduction

This paper deals with the derivation of uniform asymptotic expansions of a simple sum of essentially geometric type, which arises in the investigation of a certain parameter of the well-known slotted ALOHA collision resolution algorithm. Such algorithms are necessary for computer networks based on random multiple access broadcast channels: A number of stations (i.e., transmitting/receiving units) share a single communication channel. Data are sent in form of packets without any centralized channel arbitration mechanism. Hence, a distributed algorithm for resolving conflicts arising from simultaneous transmission attempts of multiple stations is needed.

The whole subject came up with the developement of the *ALOHA* system at the University of Hawaii in the late 1970's. Since this time, a number of varieties of the original ALOHA algorithm and, most important, a family of *tree algorithms* have been proposed, which offer better characteristics, e.g., average packet throughput; cf. [**2**] for an overview. A well-known variety is the *slotted ALOHA* algorithm, which works as follows: If a station has been involved in a collision, it transmits its packet in each subsequent slot with a fixed probability p until a successful transmission of the packet occurs. Packets are assumed to have fixed size and fit into exactly one slot. Note that a collision causes the destruction of all packets involved, hence may be detected by all stations via certain checksumming methods.

An important parameter of such an algorithm is the length of a *collision resolution interval* (especially the *average CRI-length* L_n), which is the number of slots necessary for resolving an initial collision of n transmitters when packets generated during the resolution process are not considered. Note that the CRI-length is independent of the underlying packet generating process. This parameter is well-known from the troughput analysis of certain tree algorithms (cf. [**1**] for a nice survey) and allows a significant estimation of the performance of a collision resolution algorithm. However, we should mention that the usual analysis of ALOHA algorithms is based on queueing theory, cf. [**2**] for an introduction.

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Our primary objective is not at all the straightforward derivation of the exact value of $L_n = L_{n,p}$ for the slotted ALOHA algorithm, but rather the computation of uniform asymptotic expansions of $L_{n,p}$ which are valid for reasonable values of p. As we shall see, it would be merely an exercise for undergraduate students to derive uniform asymptotic expansions for very small p (say, $p < 1/n$) and very large p (say, $p > \varepsilon$), respectively. But, simple considerations reveal that the optimal choice of p for a given n, i.e., that value of p which provides the smallest CRI-length when resolving an initial collision of multiplicity n, is approximately $p = C/n$ for some positive constant C.

However, enlarging the region of uniform validity requires some effort. First, by means of a direct asymptotic method based on the analysis of a certain sum involving binomial coefficients we shall derive a uniform expansion valid for $p \leq n^{-0.51}/2e$, $n \to \infty$. Note that our major development was the somewhat surprising relation between $L_{n,p}$ and the sum mentioned above. Second, adopting some well-established methods relying on complex analysis (cf. [3]) it is possible to obtain another expansion of $L_{n,p}$, uniformly valid for $p \geq n^{-0.99}$ as $n \to \infty$.

2. Preliminaries and Results

According to the slotted ALOHA collision resolution algorithm, we have n stations transmitting with probability p in the slot following the initial collision slot of multiplicity $n \geq 2$. The probability that exactly one of them transmits, thus decrementing the number of stations concerned by one, is $np(1-p)^{n-1}$. With the complement probability, their number remains unchanged.

Thus, the resolution process of a collision of multiplicity n may be represented by a finite automaton with $n+1$ states $\mathcal{Q}_k$, $0 \leq k \leq n$, each representing a collision of multiplicity k. $\mathcal{Q}_0$ denotes the terminal state. A state transition corresponds to a slot and is marked by the probability of the occurrence of the transition. Each path from the initial to the terminal state corresponds to a possible resolution, and its probability is the product of the markings of the transitions concerned.

This situation translates into a recurrence relation for the appropriate *probability generating functions* (PGF)

$$Q_n(z) = \sum_k q_{n,k} z^k,$$

where $q_{n,k}$ denotes the probability that the CRI-length of an initial collision of multiplicity n equals k slots. We obtain

$$\begin{aligned} Q_n(z) &= (1 - np(1-p)^{n-1}) z Q_n(z) + np(1-p)^{n-1} z Q_{n-1}(z) \qquad \text{for } n \geq 2 \\ Q_1(z) &= z. \end{aligned}$$

Obviously, $Q_j(z)$ represents the state $\mathcal{Q}_j$, where exactly j packets are waiting for transmission; the second equation handles the terminal transition, e.g., the case where exactly one packet is waiting for transmission. Solving the system of equations, we find

$$Q_n(z) = z^n \prod_{j=2}^{n} \frac{jp(1-p)^{j-1}}{1 - z(1 - jp(1-p)^{j-1})}.$$

Differentiating $Q_n(z)$ w.r.t. z using the logarithmic derivative and substituting $z = 1$, we obtain the desired expectation of the CRI-length, namely

$$L_{n,p} = 1 + \sum_{j=2}^{n} \frac{1}{pj(1-p)^{j-1}}.$$

Our major goal is the derivation of an asymptotic expression for the quantity $L_{n,p}$ as n gets large, uniformly valid for reasonable values of p. The sum above is very sensitive w.r.t. the retransmission probability p: For small values of p it is clear that $L_{n,p}$ is asymptotically equivalent to H_n/p with $H_n = \log n + O(1)$ denoting the Harmonic numbers. On the other hand, for large values of p, we may expect an exponential growth of the sum. Therefore, we have to divide the investigations in two parts, (1) for small values of p, where we use a direct method, and (2) for large values of p, were we apply a generating function method. Our major result, proved in the following section, is

THEOREM 2.1. *For $0 < p < 1$ and $n \geq 2$, the sum*

$$L_{n,p} = 1 + \sum_{j=2}^{n} \frac{1}{pj(1-p)^{j-1}}$$

has the uniform asymptotic expansions

$$L_{n,p} = \begin{cases} \dfrac{H_n - 1}{p} - H_n + \dfrac{1}{p}\displaystyle\sum_{j\geq 1} \frac{(np)^j}{jj!} + O(e^{np}) & \text{for } p \leq n^{-0.51}/2e \\[2ex] \dfrac{1}{np^2(1-p)^{n-1}}\left(1 + O\left(\dfrac{\log^4 n}{np}\right)\right) & \text{for } p \geq n^{-0.99} \end{cases}$$

as $n \to \infty$.

Those results require some additional remarks:

(1) The uniform validity of the first formula may be extended to $p \leq n^{-0.51}/a$ for some fixed $0 < a < \infty$. We choose $a = 2e$, because this ensures all our inequalities valid for all $n \geq 2$ and $p \to 0$. This restriction is not necessary as n gets large.

(2) The infinite sum in the first formula is related to an exponential integral by

$$\mathrm{Ei}(x) = \gamma + \log x + \sum_{j\geq 1} \frac{x^j}{jj!} \quad ,$$

see [4, p.228] for additional informations. Simple comparisons with the function e^x yield the estimation

$$\sum_{j\geq 1} \frac{x^j}{jj!} = \frac{e^x - x - 1}{x}\theta(x) \qquad \text{with } 1 \leq \theta(x) \leq 2.$$

Hence we may expect that the optimal retransmission probability is $p = q(n)/n$ with $q(n)$ a slowly increasing function like $\log\log n$ (necessarily $o(\log n)$), causing a minimal $L_{n,p} \approx O(\frac{n \log n}{\log\log n})$. This is an approximate asymptotic lower bound for the expected CRI-length of any controlled ALOHA algorithm, which estimates the multiplicity of the initial collision and adjusts the retransmission probability.

(3) For all values of p, we have the uniform bound

$$L_{n,p} = O(\frac{\log n}{np^2(1-p)^{n-1}}) \quad ,$$

which is already established for $p \geq n^{-0.99}$ by Theorem 2.1. For other values of p, we use the substitution $p = t/n$ with $t \leq n^{0.01}$ and the well-known relation $(1-t/n)^n \leq e^{-t}$, which implies $e^{np} = O(1/(1-p)^{n-1})$. Remembering the estimation in remark (2) provides the bound for the sum; the other terms are trivial.

(4) The O(.)-term in the second formula is very large, even for large n. The results of an elaborate computer simulation showed indeed a very good approximation via the major term of the first formula, but a weak one via the second.

3. Analysis

We start our treatment for small p by investigating a quantity $h_{n,p}$, which we found accidentially by expanding $L_{n,p}$ in powers of p:

$$h_{n,p} = \sum_{j\geq 1} \binom{j+n}{j} \frac{p^j}{j}$$

Using the fundamental recurrence of the binomial coefficients, we obtain

$$\begin{aligned} h_{n,p} &= \sum_{j\geq 1} \binom{j+n-1}{j} \frac{p^j}{j} + \sum_{j\geq 1} \binom{j+n-1}{j-1} \frac{p^j}{j} \\ &= h_{n-1,p} + 1/n \sum_{j\geq 1} \binom{j+n-1}{j} p^j = h_{n-1,p} + \frac{1}{n(1-p)^n} - 1/n \\ &= h_{0,p} + \sum_{j=1}^{n} \frac{1}{j(1-p)^j} - H_n = -\log(1-p) - H_n + \sum_{j=1}^{n} \frac{1}{j(1-p)^j}. \end{aligned}$$

We may rewrite our desired quantity in terms of $h_{n,p}$, e.g.,

$$L_{n,p} = 1 - \frac{1}{p} + \frac{1-p}{p}(h_{n,p} + H_n + \log(1-p)).$$

Defining a fixed $\varepsilon < 1/2$, we restrict ourselves to the case $p \leq n^{\varepsilon-1}/2e$. For $j \leq n^{\varepsilon}$, we have

$$\binom{n+j}{j} = \frac{n^j}{j!} \prod_{i=1}^{j} (1 + \frac{i}{n}) = \frac{n^j}{j!}(1 + O(j^2/n)) \quad ,$$

where we used the fact

$$\log \prod_{i=1}^{j}(1+\frac{i}{n}) = \sum_{i=1}^{j}\log(1+\frac{i}{n}) = O(j^2/n).$$

Now we divide the sum for $h_{n,p}$ into two parts. First, we treat the sum for $1 \le j \le n^\varepsilon$, which yields the main contribution.

$$\begin{aligned}
\sum_{j=1}^{n^\varepsilon}\binom{n+j}{j}\frac{p^j}{j} &= \sum_{j=1}^{n^\varepsilon}\frac{(np)^j}{jj!} + O\Big(1/n\sum_{j=1}^{n^\varepsilon}\frac{(np)^j}{(j-1)!}\Big)\\
&= \sum_{j=1}^{n^\varepsilon}\frac{(np)^j}{jj!} + O\Big(p\sum_{j=0}^{n^\varepsilon-1}\frac{(np)^j}{j!}\Big)\\
&= \sum_{j=1}^{n^\varepsilon}\frac{(np)^j}{jj!} + O(pe^{np})
\end{aligned}$$

We may extend the upper limit of the previous sum to infinity, as can be seen from

$$\begin{aligned}
\sum_{j\ge n^\varepsilon}\frac{(np)^j}{jj!} &= \frac{(np)^{n^\varepsilon}}{n^\varepsilon!}\Big(\frac{1}{n^\varepsilon}+\frac{np}{(n^\varepsilon+1)(n^\varepsilon+1)}+\frac{(np)^2}{(n^\varepsilon+2)(n^\varepsilon+1)(n^\varepsilon+2)}+\cdots\Big)\\
&\le \frac{(np)^{n^\varepsilon}}{n^\varepsilon n^\varepsilon!}\Big(1+\frac{np}{n^\varepsilon}+\frac{(np)^2}{(n^\varepsilon)^2}+\cdots\Big) = \frac{(np)^{n^\varepsilon}}{n^\varepsilon n^\varepsilon!}\frac{1}{1-pn^{1-\varepsilon}}\\
&= O\Big(\frac{(n^{1-\varepsilon}pe)^{n^\varepsilon}}{n^{3\varepsilon/2}}\Big) = O\big(pn^{1-2\varepsilon}(n^{1-\varepsilon}pe)^{n^\varepsilon-1}\big) \quad ,
\end{aligned}$$

where we used Stirling's expansion for the factorials in its weakest form. Because of the restriction $p \le n^{\varepsilon-1}/2e$, the last term has lower order than $O(pe^{np})$, so we may discard it. Now we investigate the second part of the sum $h_{n,p}$, which disappears too. We have

$$\begin{aligned}
S &= \sum_{j\ge n^\varepsilon}\binom{n+j}{j}\frac{p^j}{j}\\
&= \frac{(n+n^\varepsilon)\cdots(n+1)p^{n^\varepsilon}}{n^\varepsilon!}\Big(\frac{1}{n^\varepsilon}+\frac{(n+n^\varepsilon+1)p}{(n^\varepsilon+1)(n^\varepsilon+1)}+\frac{(n+n^\varepsilon+1)(n+n^\varepsilon+2)p^2}{(n^\varepsilon+2)(n^\varepsilon+1)(n^\varepsilon+2)}+\cdots\Big)\\
&\le \frac{(n+n^\varepsilon)(n+n^\varepsilon-1)\cdots(n+1)p^{n^\varepsilon}}{n^\varepsilon n^\varepsilon!}\frac{1}{1-(n^{1-\varepsilon}+1)p} \quad ,
\end{aligned}$$

where we used the fact

$$\frac{n+n^\varepsilon+k}{n^\varepsilon+k}p = (\frac{n}{n^\varepsilon+k}+1)p \le (n^{1-\varepsilon}+1)p < 1.$$

Mentioning $n + n^\varepsilon < 1.9n$ for $n \geq 2$, we find

$$S \leq \frac{(1.9np)^{n^\varepsilon}}{n^\varepsilon n^\varepsilon!}\frac{1}{1-p(n^{1-\varepsilon}+1)} = O\Big(\frac{(1.9n^{1-\varepsilon}pe)^{n^\varepsilon}}{n^{3\varepsilon/2}}\Big) = O\big(pn^{1-2\varepsilon}(1.9n^{1-\varepsilon}pe)^{n^\varepsilon-1}\big).$$

Similar to the previous case, this term is of lower order than $O(pe^{np})$. Now we have completed the estimations and obtain

$$L_{n,p} = \frac{1-p}{p}\sum_{j\geq 1}\frac{(np)^j}{jj!} + \frac{1-p}{p}H_n + \frac{1-p}{p}\log(1-p) + 1 - \frac{1}{p} + O(e^{np}).$$

The result as stated in Section 2 is derived by considering

$$\frac{1-p}{p}\log(1-p) = O(1) = O(e^{np}) \qquad \text{and}$$
$$\sum_{j\geq 1}\frac{(np)^j}{jj!} \leq \sum_{j\geq 1}\frac{(np)^j}{j!} = O(e^{np}).$$

The last problem is to find an asymptotic expression for

$$L_{n,p} = 1 + \sum_{j=2}^{n}\frac{1}{pj(1-p)^{j-1}}$$

as n gets large and p is relatively large. We use the fact

$$\sum_{j=1}^{n}\frac{1}{j(1-p)^{j-1}} = (1-p)^{1-n}\sum_{j=1}^{n}\frac{(1-p)^{n-j}}{j} = (1-p)^{1-n}f_{n,p}$$

with $f_{n,p}$ denoting the n-th Taylor-coefficient of the generating function

$$F_p(z) = \frac{-\log(1-z)}{1-(1-p)z}.$$

Following the method proposed in [3], we express $f_{n,p}$ via Cauchy's formula using a contour α composed of a circle-segment β of radius 2 around 0, with a "$\subset$-notch" γ along the positive real axis, consisting of two horizontal segments and a semicircle with radius $1/n$ around 1. Obviously, we use the branch of $\log z$, where the function is real-valued when the argument is negative.

First, from Cauchy's inequality, it is clear that the contribution from β is exponentially small in n, thus we have to investigate the main term coming from γ. We use the substitution $z = 1 + t/n$ and therefore $dz = dt/n$, where t lies on a (negatively oriented) contour Γ

consisting of two horizontal segments $\Im(t) = \pm 1$ and $0 \le \Re(t) \le n$, and a semicircle with radius 1 around 0. We obtain for some fixed $R < 2$

$$\begin{aligned} f_{n,p} &= \frac{1}{2\pi i} \int_\alpha F(z) z^{-n-1}\, dz \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{\log(-n/t)}{p-(1-p)t/n} (1+t/n)^{-n-1} \frac{dt}{n} + O(R^{-n}) \\ &= \frac{\log n}{np} I_0 - \frac{1}{np} I_1 + O(R^{-n}) \end{aligned}$$

with the abbreviation

$$I_l = \frac{1}{2\pi i} \int_\Gamma \frac{\log^l(-t)}{1-(1-p)t/np} (1+t/n)^{-n-1}\, dt.$$

In order to obtain an asymptotic expression of the integral, we choose the contour Γ_1 to be the part of Γ with the property $|t| \le \log^2 n$. We replace Γ by Γ_1 and estimate the error-term by mentioning

$$\begin{aligned} \left|(1+t/n)^{-n-1}\right| &\le (1+\log^2 n/n)^{-n-1} = e^{-\log^2 n}(1+O(\log^4 n/n)) = O(n^{-\log n}) \\ |\log^l(-t)| &= O(\log^l n) \end{aligned}$$

for t on $\Gamma_2 = \Gamma - \Gamma_1$. Moreover, because $(1-p)/p$ monotonically tends to 0 as $p \to 1$, we have

$$|1-(1-p)t/np| \ge \begin{cases} 1 - |t|/3n \ge 1/2, & \text{for } p \ge 3/4 \\ \Im(1-(1-p)t/np) \ge 1/3n, & \text{for } p < 3/4 \end{cases}$$

and therefore

$$\left|\frac{1}{1-(1-p)t/np}\right| = O(n).$$

Thus, the integral along Γ_2 yields the error-term $O(n^{-\log n} n^2 \log^l n)$. Now we replace $(1+t/n)^{-n-1} = e^{-t}(1+O(t^2/n))$ by e^{-t}, which leads to

$$I_l = (1+O(\frac{\log^4 n}{n})) \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\log^l(-t)}{1-(1-p)t/np} e^{-t}\, dt + O(n^{-\log n} n^2 \log^l n).$$

Expanding the fraction of the integrand above yields

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\log^l(-t)}{1-(1-p)t/np} e^{-t}\, dt &= \frac{1}{2\pi i} \int_{\Gamma_1} \log^l(-t) e^{-t}\, dt \\ &\quad - \frac{1-p}{np} \frac{1}{2\pi i} \int_{\Gamma_1} \frac{(-t)\log^l(-t)}{1-(1-p)t/np} e^{-t}\, dt. \end{aligned}$$

The main contribution comes from the first term, which is related to Hankel's expression of the Γ-function. We extend the contour Γ_1 to a contour H of Hankel's type, e.g., coming from $+\infty$ it encircles the origin clockwise, reaching $+\infty$ again. The error-term computes to

$$|\int_{H-\Gamma_1} \log^l(-t)e^{-t}\,dt| = O\Big(\int_{x\geq\log^2 n} xe^{-x}\,dx\Big) = O(n^{-\log n}\log^2 n).$$

Treating the estimation of the second term, we need the restriction to $p \geq n^{\rho-1}$ with $\rho > 0$. Mentioning the fact $(1-p)/np \leq n^{-\rho}$, we find

$$\begin{aligned}|\int_{\Gamma_1} \frac{(-t)\log^l(-t)}{1-(1-p)t/np}e^{-t}\,dt| &\leq C|\int_{\Gamma_1}(-t)\log^l(-t)e^{-t}\,dt| \\ &\leq C|\int_H(-t)\log^l(-t)e^{-t}\,dt| + C|\int_{H-\Gamma_1}(-t)\log^l(-t)e^{-t}\,dt| \\ &= O(1).\end{aligned}$$

The $O(1)$-term comes from relating the first of the integrals above to Hankel's expression of the Γ-function. The second integral is of lower order, as can be proved by the previous estimation, too. Mentioning the negative orientation of the contour H, we obtain by using the well-known formulas of Hankel (cf. [**5**, p.244])

$$\begin{aligned}\frac{1}{2\pi i}\int_H(-t)^{-z}e^{-t}\,dt &= \frac{1}{\Gamma(z)} \\ \frac{1}{2\pi i}\int_H \log(-t)(-t)^{-z}e^{-t}\,dt &= \frac{\psi(z)}{\Gamma(z)}\end{aligned} \quad ,$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Γ-function. Putting all things together, we have

$$\begin{aligned}I_0 &= \Big(1+O\big(\frac{\log^4 n}{n}\big)\Big)O\big(\frac{1}{np}\big) = O\big(\frac{1}{np}\big) \\ I_1 &= \Big(1+O\big(\frac{\log^4 n}{n}\big)\Big)\Big(\lim_{z\to 0}\frac{\psi(z)}{\Gamma(z)} + O\big(\frac{1}{np}\big)\Big) = -1 + O\big(\frac{\log^4 n}{n}\big).\end{aligned}$$

Here we used the well-known limiting value of the fraction above. Substituting this in the expression of the $f_{n,p}$, we obtain

$$f_{n,p} = O\big(\frac{\log n}{(np)^2}\big) + \frac{1}{np} + O\big(\frac{\log^4 n}{n^2 p}\big) = \frac{1}{np} + O\big(\frac{\log^4 n}{(np)^2}\big) \quad ,$$

and the result stated in Section 2 follows from

$$L_{n,p} = 1 - \frac{1}{p} + \frac{(1-p)^{1-n}}{p}f_{n,p}.$$

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