Institut für Automation
Abt. für Automatisierungssysteme

Technische
Universität
Wien

Projektbericht Nr. 183/1-12
Juni 1990

FCFS Scheduling in a Hard Real-Time Environment under Rush-Hour Conditions

J. Blieberger, U. Schmid

Ausschnitt aus: Salvador Dali, "Die Beständigkeit der Erinnerung"

FCFS-Scheduling in a Hard Real-Time Environment under Rush-Hour Conditions

J. BLIEBERGER AND U. SCHMID[1]

Abstract. We investigate some real-time behaviour of a (discrete time) single server system with FCFS task scheduling under rush-hour conditions. The main result deals with the probability distribution of a random variable SRD(T), which describes the time the system operates without violating a fixed task service time deadline T.

Relying on a simple general probability model, asymptotic formulæ concerning the mean and the variance of SRD(T) are determined; for instance, if the average arrival rate is larger than the departure rate, the expectation of SRD(T) is proved to fulfill $E[\mathrm{SRD}(T)] = c_1 + \mathcal{O}(T^{-3})$ for $T \to \infty$, where c_1 denotes some constant. If the arrival rate equals the departure rate, we find $E[\mathrm{SRD}(T)] \sim c_2 T^i$ for some $i \geq 2$.

Keywords. real-time behaviour, FCFS scheduling, trees, probability generating functions, singularity analysis, asymptotics.

1. INTRODUCTION

In this paper, we will study some aspects concerning the real-time behaviour of a discrete time single server system with FCFS task scheduling. Instead of using queueing theory, we apply a special tree approach which is well-known from the analysis of data structures, see [KN1], [KN3], [VF] for a survey and [BS], [SB] for another application of this approach.

We consider a system containing a task scheduler, a task list of (potential) infinite capacity, and a single server. Tasks arriving to the system are taken by the scheduler and placed into the task list according to the scheduling strategy. The server always executes the task at the head of the list, thus scheduling is done by rearranging the task list. A dummy task will be generated by the scheduler, if the list becomes empty. If the server executes a dummy task, the system is called *idle*, otherwise *busy*.

Rearranging the task list is assumed to occur at discrete points on the time axis only. The (constant) time interval between two such points is called a *cycle*. Due to this assumption, we are able to model tasks formed by indivisible, i.e., atomic actions with duration of 1 cycle. A task may need an arbitrary number of actions to execute for its completion, the dummy task as mentioned above is supposed to consist of a single no-operation action (1 cycle).

Obviously, the time axis is covered by *busy periods*, which are supposed to include the initial idle cycle, too. This definition implies the correspondence of an idle cycle and a busy period with duration of 1 cycle.

We assume that each task has associated with it a fixed *deadline* T, i.e., the task has to complete its execution within T cycles, otherwise he violates the deadline, which may cause severe problems in a hard real-time system. In order to investigate real-time performance, we are going to study the random variable *successful run duration* SRD(T) which can be

[1] Department of Automation (183/1) at the Technical University of Vienna, Treitlstraße 3/4, A-1040 Vienna

described as follows: Starting from an idle cycle, a sequence of nonviolating busy periods followed by a busy period containing at least one deadline violation is called a *run*, the sequence without the last (violating) busy period is refered to by *successful run*. The random variable SRD(T) is the length of a successful run, i.e., the time interval from the beginning of the initial idle cycle to the beginning of the (idle) cycle initiating the busy period containing the first violation of a task's deadline T.

Different scheduling strategies may be compared via the distribution of this quantity, even if the arrival process is modeled very simple (as we did). We assume an arrival process, which provides an arbitrary distributed number of task arrivals within a cycle, independent from the arrivals in the preceding cycles, and independent from the task execution times, too. The arbitrary distributed (but independent) *task execution time* is the number of cycles necessary for processing the task to completion if it would occupy the server exclusively.

In [BS] and [SB] we have studied certain scheduling algorithms in the case where the average arrival rate is smaller than the departure rate of the system. We call this case the *normal case*. In this case the system is stable, i.e., it is able to cope with the arriving tasks without forming an "unresolvable" backlog. The small arrival rate results in an exponential growth of the mean of SRD(T). If, however, the arrival rate increases, we found that the mean of SRD(T) decreases, but our former results are only valid as long as the arrival rate is smaller than the departure rate.

In this paper we are going to investigate two cases, the case where the arrival rate is larger than the departure rate, and the case where the arrival rate equals the departure rate. Both cases may be characterized by causing a high load of the system, which is known as *rush-hour conditions* (cf. [**KL**]). This time we are not able to derive the limiting distribution of SRD(T) as we have done in the normal case (cf. [**DS**]). Nevertheless, in both cases we will derive the mean and the variance of the random variable SRD(T).

NOTATIONAL REMARKS:

(1) We denote by $[z^n]f(z)$ the nth coefficient in the (formal) power series $f(z)$.
(2) We write $f(x) = \mathcal{O}(g(x))$ for $x \to x_0$ if there exists some real constant $C > 0$ independent of x which guarantees $|f(x)| \leq C|g(x)|$ for all x in a suitable neighbourhood of x_0.
(3) We use the notation $f(x) \sim g(x)$ for $x \to x_0$ if $\lim_{x \to x_0} f(x)/g(x) = 1$.

2. PROBABILITY MODEL

This section introduces the probability model used for our subsequent investigations. We assume arbitrary but independent probability distributions of both the number of task arrivals within a cycle and task execution times.

The probability generating function (PGF) of the number of task arrivals during a cycle is denoted by

$$A(z) = \sum_{k \geq 0} a_k z^k$$

and should meet the constraint $a_0 = A(0) > 0$, i.e., the probability of no arrivals during a slot should be greater than zero. This assures the existence of idle cycles. The definition implies the independence of arrivals within two arbitrary different cycles.

The PGF of the task execution time (measured in cycles) is denoted by

$$L(z) = \sum_{k \geq 0} l_k z^k$$

with the additional assumption $L(0) = 0$, i.e., the task execution time should be greater than or equal to one cycle. Again, this definition implies task execution times both independent from each other and from the arrival process. Since we are studying non-preemptive FCFS scheduling, we may deal with the overall service time, i.e., the number of cycles induced by arrivals within a cycle, instead of using the number of arrivals and corresponding service times separately. Using the property that the PGF of a sum of independent random variables is the product of the corresponding PGFs, we obtain

$$P(z) = \sum_{k \geq 0} p_k z^k = A(L(z)).$$

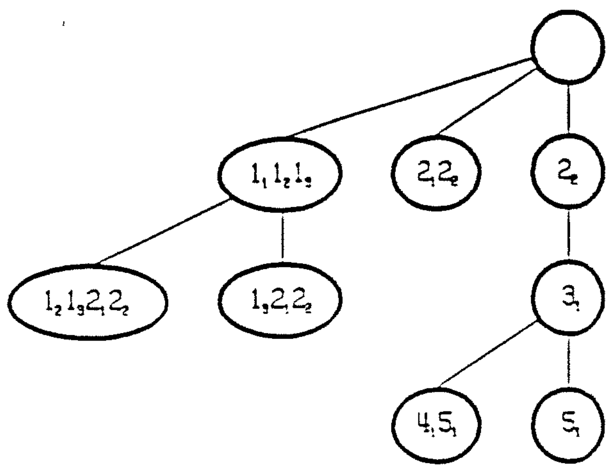
3. Tree Approach

We start our treatment by introducing an arrival sequence $\{a_n\}, n \geq 0$, where $a_n \geq 0$ counts the number of cycles caused by task arrivals during the nth busy cycle following the initial (idle!) cycle. We will establish a one-to-one mapping between arrival sequences and a family of planted planar trees, which provides a nice correspondence between deadline constraints and limited widths of the tree. Due to this fact, we may relate the original problem of investigating the random variable SRD(T) to a counting problem regarding a special (sub)family $\mathcal{B}_T$ of trees.

Let us start with an example; consider the arrival sequence

$$(3, 2, 0, 0, 0, 1, 2, 0, 0)$$

and the corresponding tree



Each vertex corresponds to a cycle n; the number of successors of a vertex equals a_n, the number of (busy) cycles caused by arrivals during the cycle; the root corresponds to the initial idle cycle 0. The execution sequence is related to the preorder traversal policy (left to right) of the tree. The 'aligned' representation of the tree above will be useful in establishing the deadline property mentioned above.

For convenience, each vertex is labeled by an expanded string representation of the task list at the beginning of the corresponding cycle, i.e., by all cycles currently forming the task list. The kth cycle of the nth task is denoted by n_k. New cycles are attached at the end of the string, the cycle actually executed is removed at the front of it. Note, however, that construction and reconstruction of tree and arrival sequence, respectively, does not depend on this labeling.

Looking carefully at our example, one obtains that the number of cycles forming the task list for all vertical aligned vertices is equal; and this is in fact true for all such trees due to the construction principle. This number represents the time interval (measured in cycles) until completion of the last cycle in the list; hence limiting the service times of the tasks by a deadline T is reflected by limiting the width of the tree to T vertices!

To obtain the connection to our probability model, we simply have to attach weights to all vertices. The weight of each vertex is equal to the probability that the vertex has its specific number of successors. The ordinary generating function (OGF) of this special family $\mathcal{B}_T$ of trees is the PGF of the length of a busy period conditioned by the fact that the busy period contains no deadline violation.

4. The Rush-Hour Case

In this section we are going to study the behaviour of our system if the average arrival rate is larger than the average departure rate, which is reflected by the fact that $P'(1) > 1$.

In order to justify our computations, we will need some constraints concerning zeros of $P(z) - z$, i.e., fixed points of $P(z)$.

Considering an arbitrary PGF $P(x)$ (with $P''(x) \not\equiv 0$ and $P'(1) > 1$) w.r.t. real arguments x, we obviously observe the trivial fixed point $x = 1$. If the Taylor expansion at $x = 1$ exists and is valid in a neighbourhood of $x = 1$, we have

$$f(x) = P(x) - x = (x-1)(P'(1) - 1) + R_2(x). \tag{1}$$

Since $P(0) = p_0 > 0$ and $P(1) = 1$, we have $f(0) > 0$ and $f(1) = 0$. Furthermore (1) implies for some ε sufficiently small that

$$f(x) < 0 \quad \text{for } x \in (1, 1-\varepsilon).$$

Hence, there exists at least one zero of $f(x)$ in $(0,1)$; let the smallest one be denoted by β. Now, it is easy to show by Rouché's Theorem that there exists only one zero within a circle around 0 with radius 1 and only one zero on the circle. Note that we have from simple geometric arguments that $P'(\beta) < 1$, which forces β to be a simple zero of $f(x)$.

Thus we state the following constraints for the PGF of the number of cycles induced by arrivals within one cycle

(1) $P(0) = p_0 > 0$, i.e., it is guaranteed that our tree construction process works.
(2) The average number of cycles induced by arrivals within one cycle should be greater than one, i.e., $P'(1) > 1$.

(3) $P''(z) \not\equiv 0$, i.e., we explicitely exclude the trivial case $P(z) = p_0 + (1 - p_0)z$.
(4) The radius of convergence R_P of $P(z)$ should be sufficiently large. We assume that $R_P > 1$.

As mentioned in Section 1, a run denotes a sequence of busy periods not violating any task's deadline followed by a busy period with at least one deadline violation. Let

$$b_{k,T} = \text{prob}\{\text{Length of a non violating busy period equals k cycles}\}$$

and

$$B_T(z) = \sum_{k \geq 0} b_{k,T} z^k$$

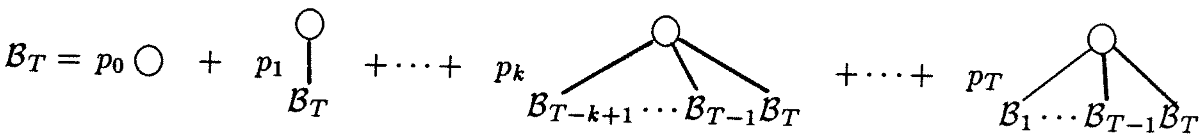
be the corresponding PGF. The PGF of the random variable SRD(T), i.e., the length of a successful run, is given by

$$S_T(z) = \sum_{k \geq 0} s_{k,T} z^k = \frac{1 - B_T(1)}{1 - B_T(z)}.$$

This follows from the fact that the PGF of the length of an arbitrary number of nonviolating busy periods is $\sum_{n \geq 0} B_T(z)^n$, and that the probability of the occurrence of the terminating violation busy period equals $1 - B_T(1)$.

In order to derive $B_T(z)$, we start with the following symbolic equation concerning our family of width constrained trees $\mathcal{B}_T$. The derivations below follow the procedure in [SB]. We decided to repeat them for the sake of completeness. Note that the family appears in the analysis of a simple register function regarding to T-ary operations, too; cf. [KP], [FRV] for details. In fact, there is a relation to the so–called left sided height of a tree.

With p_k denoting the probability of obtaining k cycles induced by arrivals within a cycle (cf. Section 2), we have



for all $T \geq 1$. According to [FL], this symbolic equation translates into a recurrence relation of the ordinary generating function

$$B_T(z) = \sum_{k=0}^{T} p_k z \prod_{j=T-k+1}^{T} B_j(z).$$

Since each vertex with k successors is weighted by $p_k z$, the coefficient of z^n in $B_T(z)$, denoted by $b_n = [z^n] B_T(z)$, is the probability of obtaining a tree with exactly n vertices. Defining

$$Q_n(z) = \frac{1}{B_n(z) \cdots B_1(z)}$$
$$Q_0(z) = 1$$

and the corresponding bivariate generating function

$$Q(s,z) = \sum_{k\geq 0} Q_k(z)s^k,$$

we obtain

$$B_T(z) = \frac{Q_{T-1}(z)}{Q_T(z)}.$$

Multiplying our fundamental recurrence relation by $Q_T(z)$ yields

$$Q_{T-1}(z) = z\sum_{k=0}^{T} p_k Q_{T-k}(z),$$

multiplying both sides by s^T and summing up for $T \geq 1$, we find

$$\begin{aligned} sQ(s,z) &= z\big(Q(s,z)P(s) - p_0\big) \\ Q(s,z) &= \frac{zp_0}{zP(s) - s}. \end{aligned}$$

The bivariate generating function $Q(s,z)$ enables us to use singularity analysis techniques for obtaining results concerning $Q_T(z)$ and $B_T(z)$, hence we are not forced to make use of explicit expressions.

We will determine the mth derivative of $Q_T(z)$, denoted by $Q_T^{(m)}(z)$, evaluated at the point $z = 1$. For practical applications, the deadline T of a task should be large compared to the duration of a cycle, hence asymptotic results for large T are satisfactory. We easily obtain

$$Q_T^{(m)}(1) = Q_T^{(m)}(z)\Big|_{z=1} = m![(z-1)^m][s^T]Q(s,z).$$

The expansion of $Q(s,z)$ at $z=1$ is found by mentioning

$$\begin{aligned} Q(s,z) &= \frac{zp_0}{zP(s) - s} \\ &= -\frac{p_0}{P(s)} \cdot \frac{(z-1)\frac{P(s)}{s-P(s)}}{1-(z-1)\frac{P(s)}{s-P(s)}} - \frac{p_0}{s-P(s)} \cdot \frac{1}{1-(z-1)\frac{P(s)}{s-P(s)}}, \end{aligned}$$

hence we are able to pick up the coefficient of $[(z-1)^m]$ directly by using the geometric series. For $m \geq 1$, we obtain

(2) $$[(z-1)^m]Q(s,z) = -\frac{p_0 s\big(P(s)\big)^{m-1}}{\big(s-P(s)\big)^{m+1}},$$

for $m = 0$, we have

(3) $$[(z-1)^0]Q(s,z) = Q(s,1) = -\frac{p_0}{s-P(s)}.$$

According to methods from singularity analysis, the coefficient of s^T is mainly determined by the singularity with smallest modulus, resulting from the denominator vanishing at this point. An overview to asymptotic methods, especially concerning the method of Darboux, may be found in [**VF**] and [**BE**]. However, we will need elementary techniques only, namely a weaker version of the so-called Cauchy's estimates. In order to derive asymptotic formulæ for the mean and the variance of SRD(T) we will use the method of *subtracting singularities*, i.e., we will locate the singularities of a corresponding generating function and use this knowledge to achieve asymptotic results. First we are going to determine the asymptotic behaviour of the mean of SRD(T)

$$\mu(T) = S_T^{(1)}(1) = \frac{B_T'(1)}{1 - B_T(1)},$$

where $B_T(z) = Q_{T-1}(z)/Q_T(z)$.

Our first step is to derive expansions of our generating functions in a suitable neighbourhood of the singularities. Thus, using

$$s - P(s) = (1 - P'(\beta))(s - \beta) + \mathcal{O}\left((s-\beta)^2\right)$$

and setting $\nu = (1 - P'(\beta))^{-1}\beta^{-1}$ we get the expansion

$$(s - P(s))^{-1} = -\nu(1 - s/\beta)^{-1} + W_1(s)$$

and derive

$$[s^T]Q(s,1) = -[s^T]\frac{p_0}{s - P(s)} = p_0\nu\beta^{-T} - [s^T]W_1(s).$$

Since $W_1(s)$ is analytic in a neighbourhood of $s = \beta$, we have to consider the singularity at $s = 1$ next, i.e., we need an expansion around $s = 1$

$$(s - P(s))^{-1} = -\frac{1}{1 - P'(1)}\frac{1}{1 - s} + W_2(s).$$

Hence we derive

$$Q_T(1) = [s^T]Q(s,1) = p_0\nu\beta^{-T} + \frac{p_0}{1 - P'(1)} + \mathcal{O}(R^{-T})$$

for some $1 < R < R_P$.

In order to derive the mean of SRD(T), we need more accurate expansions, e.g.

$$\begin{aligned} s - P(s) = &-\frac{1}{\nu}(1 - s/\beta) \\ &\times \left[1 + \gamma(1 - s/\beta) + \delta(1 - s/\beta)^2 + \varepsilon(1 - s/\beta)^3 + \mathcal{O}\left((1 - s/\beta)^4\right)\right], \end{aligned}$$

where $\gamma = \frac{1}{2}P''(\beta)\nu\beta^2$, $\delta = \frac{1}{6}P'''(\beta)\nu\beta^3$, and $\varepsilon = \frac{1}{24}P^{(4)}(\beta)\nu\beta^4$.

Thus we find

$$
(4)\qquad \begin{aligned}(s-P(s))^{-1} = &-\nu(1-s/\beta)^{-1} + \nu\gamma - \nu(\delta+\gamma^2)(1-s/\beta)\\ &+\nu(\varepsilon+2\gamma\delta+\gamma^3)(1-s/\beta)^2 + \mathcal{O}\left((1-s/\beta)^3\right)\end{aligned}
$$

and

$$
\begin{aligned}(s-P(s))^{-2} = &\ \nu^2(1-s/\beta)^{-2} - 2\nu^2\gamma(1-s/\beta)^{-1} + \nu^2\gamma^2 + 2\nu^2(\delta+\gamma^2)\\ &-\left(2\nu^2(\varepsilon+2\gamma\delta+\gamma^3) + 2\nu^2\gamma(\delta+\gamma^2)\right)(1-s/\beta) + \mathcal{O}\left((1-s/\beta)^2\right).\end{aligned}
$$

The Nth coefficient in the last formula is asymptotically given by

$$
[s^N](s-P(s))^{-2} = \nu^2(N+1)\beta^{-N} - 2\nu^2\gamma\beta^{-N} + \mathcal{O}\left(N^{-3}\beta^{-N}\right)
$$

and we obtain

$$
Q_T^{(1)}(1) = -p_0[s^{T-1}](s-P(s))^{-2} = -p_0\nu^2 T\beta^{-T+1} + 2p_0\nu^2\gamma\beta^{-T+1} + \mathcal{O}\left(T^{-3}\beta^{-T}\right),
$$

where the remainder terms are justified by a suitable *Transfer Lemma* (cf. [VF]). Now we are able to derive an asymptotic equivalent to $\mu(T) = S_T^{(1)}(1) = B_T'(1)/(1-B_T(1))$ by evaluating the denominator:

$$
\begin{aligned}1 - B_T(1) &= 1 - \frac{Q_{T-1}(1)}{Q_T(1)} = \frac{p_0\nu\beta^{-T}(1-\beta) + \mathcal{O}(R^{-T})}{p_0\nu\beta^{-T} + \mathcal{O}(1)}\\ &= 1-\beta + \mathcal{O}(R^{-T}) + \mathcal{O}(\beta^T).\end{aligned}
$$

Finally we need an asymptotic formula for the numerator, which is given by

$$
B_T'(1) = \frac{Q_{T-1}'(1)}{Q_T(1)} - \frac{Q_{T-1}(1)Q_T'(1)}{Q_T^2(1)}.
$$

We derive

$$
Q_T^{-1}(1) = \frac{1}{p_0\nu}\beta^T + \mathcal{O}(\beta^{2T})
$$

and get

$$
\begin{aligned}\frac{Q_{T-1}'(1)}{Q_T(1)} &= \left(-p_0\nu^2(T-1)\beta^{-T+2} + 2p_0\nu^2\gamma\beta^{-T+2} + \mathcal{O}(T^{-3}\beta^{-T})\right)\left((p_0\nu)^{-1}\beta^T + \mathcal{O}(\beta^{2T})\right)\\ &= -\nu\beta^2(T-1) + 2\nu\gamma\beta^2 + \mathcal{O}(T^{-3})\end{aligned}
$$

and

$$
\begin{aligned}\frac{Q_{T-1}(1)Q_T'(1)}{Q_T^2(1)} &= \left[p_0\nu\beta^{-T+1} + \frac{p_0}{1-P'(1)} + \mathcal{O}(R^{-T})\right]\\ &\quad\times\left[-p_0\nu^2 T\beta^{-T+1} + 2p_0\nu^2\gamma\beta^{-T+1} + \mathcal{O}(T^{-3}\beta^{-T})\right]\\ &\quad\times\left[\frac{1}{p_0\nu}\beta^T + \mathcal{O}(\beta^{2T})\right]^2\\ &= -\nu\beta^2 T + 2\nu\gamma\beta^2 + \mathcal{O}(T^{-3}).\end{aligned}
$$

Hence, summing up, we have shown

$$
B_T'(1) = \nu\beta^2 + \mathcal{O}(T^{-3})
$$

and are able to estimate the mean of SRD(T).

THEOREM 1. *With the notations from above, the mean of* SRD(T) *fulfills for* $T \to \infty$

$$\mu(T) = \frac{B_T'(1)}{1 - B_T(1)} = \frac{\nu\beta^2}{1-\beta} + \mathcal{O}(T^{-3}) = \frac{\beta}{1-\beta}\frac{1}{1-P'(\beta)} + \mathcal{O}(T^{-3}). \quad \blacksquare$$

In order to determine an asymptotic formula for the variance of SRD(T) we need some more expansions based on (4), e.g.

$$\begin{aligned}(s - P(s))^{-3} &= -\nu^3 (1 - s/\beta)^{-3} + 3\nu^3\gamma (1 - s/\beta)^{-2} \\ &\quad - 3\nu^3(\delta + 2\gamma^2)(1 - s/\beta)^{-1} + c_1 + \mathcal{O}(1 - s/\beta),\end{aligned}$$

where c_1 denotes a constant, and

$$\begin{aligned}P(s) &= \beta - P'(\beta)\beta (1 - s/\beta) + \frac{1}{2}P''(\beta)\beta^2 (1 - s/\beta)^2 \\ &\quad - \frac{1}{6}P'''(\beta)\beta^3 (1 - s/\beta)^3 + \mathcal{O}\left((1 - s/\beta)^4\right).\end{aligned}$$

If we set $f(s) = P(s)(s - P(s))^{-3}$, we get

$$f(s) = -\nu^3\beta (1 - s/\beta)^{-3} + f_1 (1 - s/\beta)^{-2} + g_1 (1 - s/\beta)^{-1} + c_2 + \mathcal{O}(1 - s/\beta)$$

where $f_1 = 3\nu^3\gamma\beta + \nu^3\beta P'(\beta)$, $g_1 = -3\beta\nu^3(\delta + 2\gamma^2) - 3\nu^3\gamma\beta P'(\beta) - \frac{1}{2}\nu^3\beta^2 P''(\beta)$ and c_2 is a constant, whose value could also be expressed in terms involving derivatives of $P(z)$ and β. Since, however, we will not need it for our asymptotic expansions, we do not give its exact value. The Nth coefficient of $f(s)$ fulfills

$$[s^N]f(s) = -\nu^3\beta\binom{N+2}{2}\beta^{-N} + f_1(N+1)\beta^{-N} + g_1\beta^{-N} + \mathcal{O}(N^{-2}\beta^{-N}).$$

Thus we obtain

$$\begin{aligned}Q_T^{(2)}(1) &= -2p_0[s^{T-1}]P(s)(s - P(s))^{-3} \\ &= 2p_0\nu^3\beta\binom{T+1}{2}\beta^{-T+1} - 2p_0 f_1 T\beta^{-T+1} - 2p_0 g_1\beta^{-T+1} + \mathcal{O}(T^{-2}\beta^{-T}) \\ &= p_0\nu^3 T^2\beta^{-T+2} + fT\beta^{-T+1} + g\beta^{-T+1} + \mathcal{O}(T^{-2}\beta^{-T})\end{aligned}$$

where $f = 2p_0 f_1 + p_0\nu^3\beta$ and $g = 2p_0 g_1$ for simplicity.

Our next goal is to derive asymptotic results for $B_T^{(2)}(1) = S_1 + S_2 + S_3$, where $S_1 = Q_T^{-1}(1)Q_{T-1}^{(2)}(1)$, $S_2 = 2(Q_T^{-1}(1))^{(1)}Q_{T-1}^{(1)}(1)$, and $S_3 = (Q_T^{-1}(1))^{(2)}Q_{T-1}(1)$. We get

$$\begin{aligned}S_1 &= \left((p_0\nu)^{-1}\beta^T + \mathcal{O}(\beta^{2T})\right) \\ &\quad \times \left(p_0\nu^3(T-1)^2\beta^{-T+3} + f(T-1)\beta^{-T+2} + g\beta^{-T+2} + \mathcal{O}(T^{-2}\beta^{-T})\right) \\ &= \nu^2\beta^3(T-1)^2 + \frac{f}{p_0\nu}\beta^2(T-1) + \frac{g}{p_0\nu}\beta^2 + \mathcal{O}(T^{-2}).\end{aligned}$$

Using $\frac{d}{dz}Q_T^{-1}(z) = -Q_T^{(1)}(z)/Q_T^2(z)$ we derive

$$
\begin{aligned}
S_2 = &- 2\left(-p_0\nu^2 T\beta^{-T+1} + 2p_0\nu^2\gamma\beta^{-T+1} + \mathcal{O}(T^{-3}\beta^{-T})\right) \\
&\times \left((p_0\nu)^{-1}\beta^T + \mathcal{O}(\beta^{2T})\right)^2 \\
&\times \left(-p_0\nu^2(T-1)\beta^{-T+2} + 2p_0\nu^2\gamma\beta^{-T+2} + \mathcal{O}(T^{-3}\beta^{-T})\right) \\
= &- 2\left[\nu^2\beta^3 T(T-1) - 4\nu^2\gamma\beta^3 T + 2\nu^2\gamma\beta^3 + 4\nu^2\gamma^2\beta^3 + \mathcal{O}(T^{-2})\right] \\
= &- 2\nu^2\beta^3 T^2 + (8\nu^2\gamma\beta^3 + 2\nu^2\beta^3)T - (8\nu^2\gamma^2\beta^3 + 4\nu^2\gamma\beta^3) + \mathcal{O}(T^{-2})
\end{aligned}
$$

Taking into account $\frac{d^2}{dz^2}Q_T^{-1}(z) = -Q_T^{(2)}(z)/Q_T^2(z) + 2[Q_T^{(1)}(z)]^2/Q_T^3(z)$ we see that $S_3 = S_4 + S_5$, where $S_4 = [-Q_T^{(2)}(1)/Q_T^2(1)]Q_{T-1}(1)$ and $S_5 = [2[Q_T^{(1)}(1)]^2/Q_T^3(1)]Q_{T-1}(1)$ and derive

$$
\begin{aligned}
S_4 = &- \left(p_0\nu^3 T^2\beta^{-T+2} + fT\beta^{-T+1} + g\beta^{-T+1} + \mathcal{O}(T^{-2}\beta^{-T})\right) \\
&\times \left((p_0\nu)^{-1}\beta^T + \mathcal{O}(\beta^{2T})\right)^2 \\
&\times \left(p_0\nu\beta^{-T+1} + \frac{p_0}{1-P'(1)} + \mathcal{O}(R^{-T})\right) \\
= &- \nu^2\beta^3 T^2 - \frac{f}{p_0\nu}\beta^2 T + \frac{g}{p_0\nu}\beta^2 + \mathcal{O}(T^{-2})
\end{aligned}
$$

and

$$
\begin{aligned}
S_5 = &2\left(-p_0\nu^2 T\beta^{-T+1} + 2p_0\nu^2\gamma\beta^{-T+1} + \mathcal{O}(T^{-3}\beta^{-T})\right)^2 \\
&\times \left(p_0\nu\beta^{-T+1} + \mathcal{O}(1)\right) \\
&\times \left((p_0\nu)^{-1}\beta^T + \mathcal{O}(\beta^{2T})\right)^3 \\
= &2\nu^2\beta^3 T^2 - 8\nu^2\gamma\beta^3 T + 8\nu^2\gamma^2\beta^3 + \mathcal{O}(T^{-2}).
\end{aligned}
$$

Summing up we get

$$B_T^{(2)}(1) = \eta + \mathcal{O}(T^{-2}),$$

where

$$
\begin{aligned}
\eta &= 2\frac{g}{p_0\nu}\beta^2 + \nu^2\beta^3 - 4\nu^2\gamma\beta^3 - \frac{f}{p_0\nu}\beta^2, \\
f &= p_0\nu^3\beta - 6p_0\nu^3\gamma\beta - 2p_0\nu^3\beta P'(\beta), \quad \text{and} \\
g &= 6p_0\beta\nu^3(\delta + 2\gamma^2) + 6p_0\nu^3\gamma\beta P'(\beta) + p_0\nu^3\beta^2 P''(\beta).
\end{aligned}
$$

Since

$$S_T''(1) = \frac{B_T''(1)}{1 - B_T(1)} + 2\mu(T)^2$$

and

(5) $$\mathrm{Var}(T) = S_T''(1) + \mu(T) - \mu(T)^2,$$

we can summarize our previous estimations in the following theorem.

THEOREM 2. *With the notations from above, the variance of the random variable* SRD(T) *fulfills for* $T \to \infty$

$$\mathrm{Var}(T) = \frac{\eta + \nu\beta^2}{1-\beta} + \frac{\nu^2\beta^4}{(1-\beta)^2} + \mathcal{O}(T^{-2}). \qquad \blacksquare$$

The rest of this section is devoted to an application of the preceding general formulæ to the Poisson case. Suppose

$$P(z) = e^{\lambda(z-1)},$$

the PGF of a Poisson distribution with rate λ. Note that $P'(1) = \lambda$, i.e., the rate equals the average number of cycles induced by arrivals within a cycle, and that we are mainly interested in large values of λ.

In order to derive the most critical quantity β, we have to study the zero of $P(s) - s$ in the interval $(0,1)$. This is easily done by applying the Lagrange inversion formula (cf. e.g. [**DB**]) as can be seen by writing $z = \lambda s$ and $\mu = \lambda e^{-\lambda}$ which yields

$$\mu = ze^{-z}.$$

Thus using the Lagrange inversion formula, we derive

$$z = \sum_{k\geq 1} c_k \mu^k,$$

where $c_k = k^{k-1}/k!$.

Hence $\beta = \beta(\lambda)$ is (for $\lambda \to \infty$) given by

$$\beta = \sum_{k\geq 1} \frac{(k\lambda)^{k-1}}{k!} e^{-\lambda k} = e^{-\lambda} + \mathcal{O}(\lambda e^{-2\lambda}).$$

Using this and by mentioning Theorem 1, we have shown the following corollary.

COROLLARY. *Under rush-hour conditions Poisson arrivals cause the mean of* SRD(T) *to fulfill for* $T \to \infty$

$$\mu(T) = c(\lambda) + \mathcal{O}(T^{-3}),$$

where $c(\lambda) = \dfrac{\beta}{1-\beta}\dfrac{1}{1-\lambda\beta} = e^{-\lambda} + \mathcal{O}(\lambda e^{-2\lambda})$ *for* $\lambda \to \infty$. $\blacksquare$

5. THE BALANCED CASE

In this section we are going to consider the case where average arrival and departure rates are equal, i.e., we assume

$$P(1) = P'(1) = 1$$

If the Taylor expansion exists in a neighborhood of $x = 1$, we may write for some $i \geq 2$

$$P(x) - x = \psi_i(x-1)^i + \psi_{i+1}(x-1)^{i+1} + R_{i+2}(x),$$

where $\psi_i = P^{(i)}(1)/i! \neq 0$, $\psi_{i+1} = P^{(i+1)}(1)/(i+1)!$ which may be equal to zero, and $R_{i+2}(x) = \mathcal{O}\left((x-1)^{i+2}\right)$. From our previous assumptions we may conclude that i is even, because otherwise there would exist a zero $0 < \xi < 1$ of $P(x)$, which can be seen by simple geometric arguments. This, however, would contradict our assumption that $P(z)$ has non-negative coefficients. So i has to be even, but we will not use this fact in the following treatment.

We can state the following constraints for the PGF of the number of cycles induced by arrivals within one cycle

(1) $P(0) = p_0 > 0$, i.e., it is guaranteed that our tree construction process works.
(2) The average number of cycles induced by arrivals within one cycle should be equal to one, i.e., $P'(1) = 1$.
(3) $P''(z) \not\equiv 0$, i.e., we explicitely exclude the trivial case $P(z) = p_0 + (1-p_0)z$.
(4) The radius of convergence R_P of $P(z)$ should be sufficiently large. We assume that $R_P > 1$.

Using the formulæ derived for the GFs of the moments of SRD(T) in Section 4 (cf. (2) and (3)), we need the following expansion

$$\begin{aligned}(s-P(s))^{-1} &= \frac{(-1)^{i+1}}{\psi_i}(1-s)^{-i}\left[1+\frac{\psi_{i+1}}{\psi_i}(1-s)+\mathcal{O}\left((1-s)^2\right)\right]\\ &= \frac{(-1)^{i+1}}{\psi_i}(1-s)^{-i}+(-1)^{i+1}\frac{\psi_{i+1}}{\psi_i^2}(1-s)^{-i+1}+\mathcal{O}\left((1-s)^{-i+2}\right)\end{aligned}$$

Now, we are able to look up the coefficient we are interested in

$$\begin{aligned}Q_T(1) &= -p_0[s^T](s-P(s))^{-1}\\ &= (-1)^i\frac{p_0}{\psi_i}\binom{T+i-1}{i-1}+(-1)^i\frac{p_0\psi_{i+1}}{\psi_i^2}\binom{T+i-2}{i-2}+\mathcal{O}(T^{i-3})\end{aligned}$$

Using

$$\binom{T+a}{b} = \frac{T^b}{b!}\left(1+\frac{b}{2}(1+2a-b)T^{-1}+\mathcal{O}(T^{-2})\right), \tag{6}$$

for $T \to \infty$ and for fixed values of a and b, we find

$$\begin{aligned}Q_T(1) &= \frac{(-1)^i}{(i-1)!}\frac{p_0}{\psi_i}T^{i-1}+\frac{(-1)^i}{(i-2)!}\frac{p_0}{\psi_i}\left(\frac{i}{2}+\frac{\psi_{i+1}}{\psi_i}\right)T^{i-2}+\mathcal{O}(T^{i-3})\\ &= \frac{(-1)^i}{(i-1)!}\frac{p_0}{\psi_i}T^{i-1}\left[1+(i-1)\left(\frac{i}{2}+\frac{\psi_{i+1}}{\psi_i}\right)T^{-1}+\mathcal{O}(T^{-2})\right]\end{aligned}$$

and finally

$$1-B_T(1) = \frac{i-1}{T}+\mathcal{O}(T^{-2}).$$

In order to derive an asymptotic expression for

$$B_T^{(1)}(1) = \frac{Q_{T-1}^{(1)}(1)}{Q_T(1)} - \frac{Q_{T-1}(1)Q_T^{(1)}(1)}{Q_T^2(1)} = S_1 - S_2 \tag{7}$$

we need asymptotic results for

$$Q_T^{(1)}(1) = -p_0[s^{T-1}](s - P(s))^{-2}.$$

We use the expansion

$$(s - P(s))^{-2} = \frac{1}{\psi_i^2}(1-s)^{-2i} + 2\frac{\psi_{i+1}}{\psi_i^3}(1-s)^{-2i+1} + \mathcal{O}\left((1-s)^{-2i+2}\right)$$

to derive

$$[s^N](s - P(s))^{-2} = \frac{1}{\psi_i^2}\binom{N+2i-1}{2i-1} + 2\frac{\psi_{i+1}}{\psi_i^3}\binom{N+2i-2}{2i-2} + \mathcal{O}(N^{2i-3})$$

which, mentioning (6), gives

$$Q_T^{(1)}(1) = -\frac{p_0}{\psi_i^2}\frac{1}{(2i-1)!}T^{2i-1}\left[1 + (2i-1)\left(\frac{2i-2}{2} + 2\frac{\psi_{i+1}}{\psi_i}\right)T^{-1} + \mathcal{O}(T^{-2})\right].$$

Hence we obtain

$$\begin{aligned}
S_1 &= \frac{(-1)^{i+1}}{\psi_i}\frac{(i-1)!}{(2i-1)!}T^i\left[1 + (2i-1)\left((i-2) + 2\frac{\psi_{i+1}}{\psi_i}\right)T^{-1} + \mathcal{O}(T^{-2})\right] \\
&\quad \times \left[1 + (i-1)\left(\frac{i}{2} + \frac{\psi_{i+1}}{\psi_i}\right)T^{-1} + \mathcal{O}(T^{-2})\right]^{-1} \\
&= \frac{(-1)^{i+1}}{\psi_i}\frac{(i-1)!}{(2i-1)!}T^i\left[1 + \left(\frac{3i^2-9i+4}{2} + (3i-1)\frac{\psi_{i+1}}{\psi_i}\right)T^{-1} + \mathcal{O}(T^{-2})\right]
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \frac{Q_{T-1}(1)}{Q_T(1)} \times \frac{Q_T^{(1)}(1)}{Q_T(1)} \\
&= \left(1 - (i-1)T^{-1} + \mathcal{O}(T^{-2})\right) \times S_1\left(1 + (2i-1)T^{-1} + \mathcal{O}(T^{-2})\right)
\end{aligned}$$

where the last factor takes into account that we have to substitute $T+1$ in the numerator of S_1 (cf. (7)). Hence we derive

$$\begin{aligned}
S_2 &= S_1\left(1 + iT^{-1} + \mathcal{O}(T^{-2})\right) \\
&= \frac{(-1)^{i+1}}{\psi_i}\frac{(i-1)!}{(2i-1)!}T^i\left[1 + \left(\frac{3i^2-7i+4}{2} + (3i-1)\frac{\psi_{i+1}}{\psi_i}\right)T^{-1} + \mathcal{O}(T^{-2})\right],
\end{aligned}$$

which implies

$$B_T^{(1)}(1) = \frac{(-1)^i}{\psi_i}\frac{i!}{(2i-1)!}T^{i-1}\left(1 + \mathcal{O}(T^{-1})\right).$$

Thus we can estimate the mean of SRD(T) for large T.

THEOREM 3. *An asymptotic expression for the mean of* SRD(T) *is for* $T \to \infty$ *given by*

$$\mu(T) = \frac{B_T^{(1)}(1)}{1 - B_T(1)} = \frac{(-1)^i}{\psi_i} \frac{i!}{(i-1)(2i-1)!} T^i \left(1 + \mathcal{O}(T^{-1})\right) . \quad \blacksquare$$

In order to derive asymptotic results for the variance of SRD(T) in the balanced case we need further expansions, e.g.

$$(s - P(s))^{-3} = \frac{(-1)^{i+1}}{\psi_i^3}(1-s)^{-3i}\left[1 + 3\frac{\psi_{i+1}}{\psi_i}(1-s) + \mathcal{O}\left((1-s)^2\right)\right] \quad \text{and}$$

$$P(s) = 1 - (1-s) + \mathcal{O}\left((1-s)^2\right)$$

If we let denote

$$f(s) = P(s)(s - P(s))^{-3} = \frac{(-1)^{i+1}}{\psi_i^3}(1-s)^{-3i}\left[1 + \left(3\frac{\psi_{i+1}}{\psi_i} - 1\right)(1-s) + \mathcal{O}\left((1-s)^2\right)\right]$$

$$= \frac{(-1)^{i+1}}{\psi_i^3}(1-s)^{-3i} + \frac{(-1)^{i+1}}{\psi_i^3}\left(3\frac{\psi_{i+1}}{\psi_i} - 1\right)(1-s)^{-3i+1} + \mathcal{O}\left((1-s)^{-3i+2}\right),$$

we get for the coefficient using (6)

$$[s^N]f(s) = \frac{(-1)^{i+1}}{\psi_i^3}\binom{N+3i-1}{3i-1} + \frac{(-1)^{i+1}}{\psi_i^3}\left(3\frac{\psi_{i+1}}{\psi_i} - 1\right)\binom{N+3i-2}{3i-2} + \mathcal{O}(N^{3i-3})$$

$$= \frac{(-1)^{i+1}}{\psi_i^3}\frac{N^{3i-1}}{(3i-1)!}\left(1 + 3i\frac{3i-1}{2N} + \mathcal{O}(N^{-2})\right)$$

$$+ \frac{(-1)^{i+1}}{\psi_i^3}\left(3\frac{\psi_{i+1}}{\psi_i} - 1\right)\frac{N^{3i-2}}{(3i-2)!} + \mathcal{O}(N^{3i-3})$$

$$= \frac{(-1)^{i+1}}{\psi_i^3}\frac{N^{3i-1}}{(3i-1)!}\left[1 + (3i-1)\left(3\frac{\psi_{i+1}}{\psi_i} - \frac{3i-2}{2}\right)N^{-1} + \mathcal{O}(N^{-2})\right].$$

Thus we obtain

$$Q_T^{(2)}(1) = -2p_0[s^{T-1}]f(s)$$

$$= 2p_0\frac{(-1)^i}{\psi_i^3}\frac{T^{3i-1}}{(3i-1)!}\left[1 + (3i-1)\left(3\frac{\psi_{i+1}}{\psi_i} - \frac{3i-4}{2}\right)T^{-1} + \mathcal{O}(T^{-2})\right].$$

Our next goal is to derive asymptotic results for $B_T^{(2)}(1) = U_1 + U_2 + U_3$, where $U_1 = Q_T^{-1}(1)Q_{T-1}^{(2)}(1)$, $U_2 = 2(Q_T^{-1}(1))^{(1)}Q_{T-1}^{(1)}(1)$, and $U_3 = (Q_T^{-1}(1))^{(2)}Q_{T-1}(1)$. Mentioning (7), we get

$$U_1 = \frac{2}{\psi_i^2}\frac{(i-1)!}{(3i-1)!}T^{2i} \quad \left[1 + \left(\frac{\psi_{i+1}}{\psi_i}(8i-2) + (4i^2 - 10i + 3)\right)T^{-1} + \mathcal{O}(T^{-2})\right],$$

$$U_2 = -2\frac{Q_T^{(1)}(1)Q_{T-1}^{(1)}(1)}{Q_T^2(1)} = -2S_1^2\left[1 + (2i-1)T^{-1} + \mathcal{O}(T^{-2})\right]$$

$$= -\frac{2}{\psi_i^2}\frac{(i-1)!^2}{(2i-1)!^2}T^{2i}\left[1 + \left(\frac{\psi_{i+1}}{\psi_i}(6i-2) + (3i^2 - 9i + 5)\right)T^{-1} + \mathcal{O}(T^{-2})\right], \quad \text{and}$$

$$U_3 = U_4 + U_5,$$

where $U_4 = -Q_T^{(2)}(1)Q_{T-1}(1)/Q_T^2(1)$ and $U_5 = 2[Q_T^{(1)}(1)]^2Q_{T-1}(1)/Q_T^3(1)$.

Mentioning

$$U_4 = -\frac{Q_{T-1}(1)}{Q_T(1)}U_1\left(1+(3i-1)T^{-1}+\mathcal{O}(T^{-2})\right),$$

where again we append a correcting factor in order to use the already estimated term U_1, we derive

$$\begin{aligned}
U_4 &= -\left(1-(i-1)T^{-1}+\mathcal{O}(T^{-2})\right)U_1\left(1+(3i-1)T^{-1}+\mathcal{O}(T^{-2})\right)\\
&= -U_1\left(1+2iT^{-1}+\mathcal{O}(T^{-2})\right)\\
&= -\frac{2}{\psi_i^2}\frac{(i-1)!}{(3i-1)!}T^{2i}\left[1+\left(\frac{\psi_{i+1}}{\psi_i}(8i-2)+(4i^2-8i+3)\right)T^{-1}+\mathcal{O}(T^{-2})\right]
\end{aligned}$$

and similarily

$$\begin{aligned}
U_5 &= -\frac{Q_{T-1}(1)}{Q_T(1)}U_2\left(1+(2i-1)T^{-1}+\mathcal{O}(T^{-2})\right)\\
&= -U_2\left(1+iT^{-1}+\mathcal{O}(T^{-2})\right)\\
&= \frac{2}{\psi_i^2}\frac{(i-1)!^2}{(2i-1)!^2}T^{2i}\left[1+\left(\frac{\psi_{i+1}}{\psi_i}(6i-2)+(3i^2-6i+3)\right)T^{-1}+\mathcal{O}(T^{-2})\right].
\end{aligned}$$

Thus we derive

$$B_T^{(2)}(1) = \frac{2}{\psi_i^2}T^{2i-1}\left[\frac{(i-1)!^2}{(2i-1)!^2}i - \frac{(i-1)!}{(3i-1)!}2i\right]\left(1+\mathcal{O}(T^{-1})\right).$$

Using the expression for $\mathrm{Var}(T)$ (cf. (5)), we have shown the following theorem.

THEOREM 4. *With the notations from above, the variance of the random variable* SRD(T) *is for* $T \to \infty$ *given by*

$$\mathrm{Var}(T) = \frac{1}{\psi_i^2(i-1)}T^{2i}\left[2i\frac{(i-1)!^2}{(2i-1)!^2} - 4i\frac{(i-1)!}{(3i-1)!} + \frac{1}{i-1}\frac{i!^2}{(2i-1)!^2}\right]\left(1+\mathcal{O}(T^{-1})\right). \quad \blacksquare$$

REFERENCES

BE. E.A. Bender, *Asymptotic methods in enumeration*, SIAM Review **16** (1974), 485–515.

BS. J. Blieberger, U. Schmid, *Some Investigations on Preemptive LCFS Scheduling in Hard Real-Time Applications*, (1991) (to appear).

DB. N. G. de Bruijn, "Asymptotic Methods in Analysis," Dover Publications, Inc., New York, 1981.

DS. M. Drmota, U. Schmid, *Exponential Limiting Distributions in Queuing Systems with Deadlines*, (1991) (to appear).

FL. P. Flajolet, "Analyse d'algorithmes de manipulation d'arbes et de fichiers," Thèse Paris-Sud-Orsay, 1979; Cahiers de BURO **34-35** (1981), 1–209.

FRV. Ph. Flajolet, J. C. Raoult, J. Vuillemin, *The number of registers required to evaluate arithmetic expressions*, Theoret. Comput. Sci. **9** (1979), 99–125.

KP. P. Kirschenhofer, H. Prodinger, *On the Recursion Depth of Special Tree Traversal Algorithms*, Information and Computation **74** (No. 1, July 1987), 15–32.

KL. L. Kleinrock, "Queueing Systems. Vol. 1 and Vol. 2," John Wiley, New York, 1975.

KN1. D.E. Knuth, "The Art of Computer Programming, Volume 1: Fundamental Algorithms. 2nd ed.," Addison-Wesley, Reading, Mass., 1973.

KN3. ———, "The Art of Computer Programming, Volume 3: Sorting and Searching.," Addison-Wesley, Reading, Mass., 1973.

SB. U. Schmid, J. Blieberger, *Some Investigations on FCFS Scheduling in Hard Real-Time Applications*, Journal of Computer and System Sciences (1991) (to appear).

VF. J.S. Vitter, P. Flajolet, *Average-Case Analysis of Algorithms and Data Structures*, Handbook of Theoretical Computer Science (J. van Leeuwen, ed.) (1990), North-Holland.