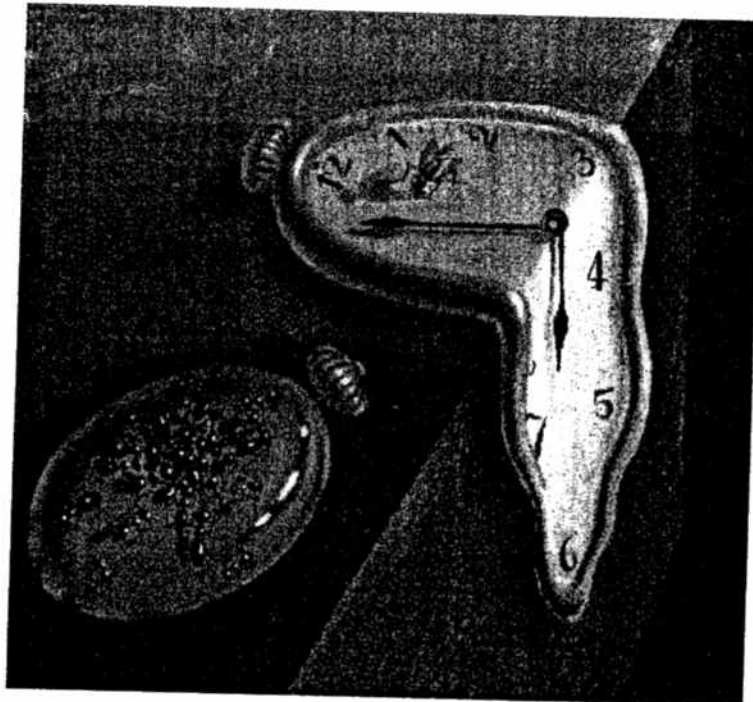


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**On non-preemptive LCFS Scheduling with Deadlines**  
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# On Nonpreemptive LCFS Scheduling with Deadlines

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**Abstract.** We investigate some real time behaviour of a (discrete time) single server system with nonpreemptive LCFS task scheduling. The main results deal with the probability distribution of a random variable  $\text{SRD}(T)$ , which describes the time the system operates without any violation of a fixed task service time deadline  $T$ . A tree approach, similar to those already used for the derivation of the same quantities for other scheduling disciplines (e.g., FCFS) is suitable here again, establishing the power of such techniques once more.

Relying on a simple general probability model, asymptotic formulas concerning all moments of  $\text{SRD}(T)$  are determined; for example, the expectation of  $\text{SRD}(T)$  is proved to grow exponentially in  $T$ , i.e.,  $E[\text{SRD}(T)] \sim CT^{3/2}\rho^T$  for some  $\rho > 1$ . Our computations rely on a multivariate (asymptotic) coefficient extraction technique which we called asymptotic separation.

## 1. BASICS

In this paper we shall study some aspects concerning the real time behaviour of a discrete time single server system with nonpreemptive LCFS task scheduling. Instead of using queueing theory, we apply a special tree approach already used for the derivation of similar results in the case of FCFS and preemptive LCFS scheduling, see [2], [8]. Both papers contain a very detailed introduction to the model, too.

The outline of the paper is as follows: After a short description of the underlying model and some questions of interest, we provide the tree approach suitable for the combinatorial and asymptotic computations in Section 3 and 4. Section 5 is devoted to our final results. Some conclusions are appended in Section 6.

We consider a system containing a task scheduler, a task list and a single server. Tasks arriving to the system are taken by the scheduler and placed into the task list according to the scheduling strategy. The server always executes the task at the head of the list, thus scheduling is done by rearranging the task list. A dummy task will be generated by the scheduler if the list becomes empty. If the server executes a dummy task the system is called *idle*, otherwise *busy*.

Rearranging the task list is assumed to occur at discrete points on the time axis only, without any overhead. The (constant) time interval between two such points is called a *cycle*. Due to this assumption we are able to model tasks formed by indivisible (atomic) actions with duration of 1 cycle. The *task execution time* of a task is the number of cycles necessary for processing the task to completion if it might occupy the server exclusively. A 'regular' task may have an arbitrary task execution time, a dummy task as mentioned above is supposed to consist of a single no-operation action (1 cycle). The *service time* of a task is the time (measured in cycles) from the beginning of the cycle in which the corresponding task arrives at the system to the end of the last cycle of that task.

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Obviously, the time axis is covered by *busy periods*, which are supposed to include the initial idle cycle, too. Note, that this definition implies the correspondence between an idle cycle and a busy period with duration of 1 cycle. A sequence of busy periods without any violation of any task's service time deadline, followed by a busy period containing at least one deadline violation is called a *run*, the sequence without the last (violating) busy period is referred to by *successful run*.

In order to investigate real time performance, we shall study the *successful run duration*  $SRD(T)$ , which is the time interval from the beginning of the initial idle cycle to the beginning of the (idle) cycle initiating the busy period containing the first violation of a task's deadline  $T$ .

We assume an arrival process which provides an arbitrary distributed number of task arrivals within a cycle, independent from the arrivals in the preceding cycles, and independent from the arbitrary distributed task execution times, too.

The probability generating function (PGF) of the number of task arrivals during a cycle is denoted by

$$A(z) = \sum_{k \geq 0} a_k z^k \quad (1.1)$$

and should meet the constraint  $a_0 = A(0) > 0$ , i.e., the probability of no arrivals during a cycle should be greater than zero. This assures the existence of idle cycles.

The PGF of the task execution time (measured in cycles) is denoted by

$$L(z) = \sum_{k \geq 0} l_k z^k \quad (1.2)$$

with the additional assumption  $L(0) = 0$ , i.e., the task execution time should be greater than or equal to one cycle. Note, that we assume an a priori knowledge of the task execution time at the time the task arrives.

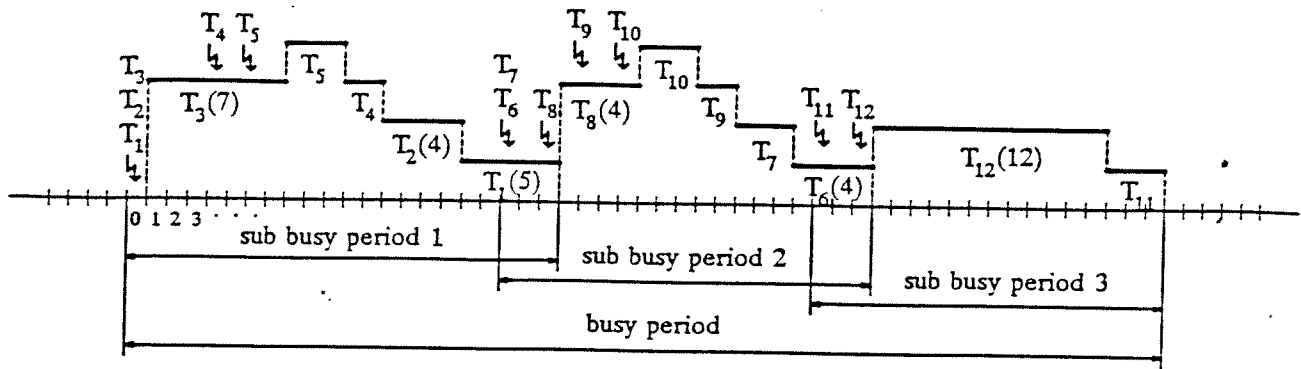
For technical reasons we shall need some additional conditions concerning the behaviour of  $P(z) = A(L(z))$ , which are summarized in Section 4.

We should mention that the number of probability distributions meeting our constraints is considerably limited due to the required independency. An example for a suitable model is based on an interarrival distribution with the so-called memoryless property, i.e., an exponential or geometric distribution, leading to (well-thumbed) Poisson- or Bernoulli-type arrivals within a cycle, respectively.

## 2. TREE APPROACH

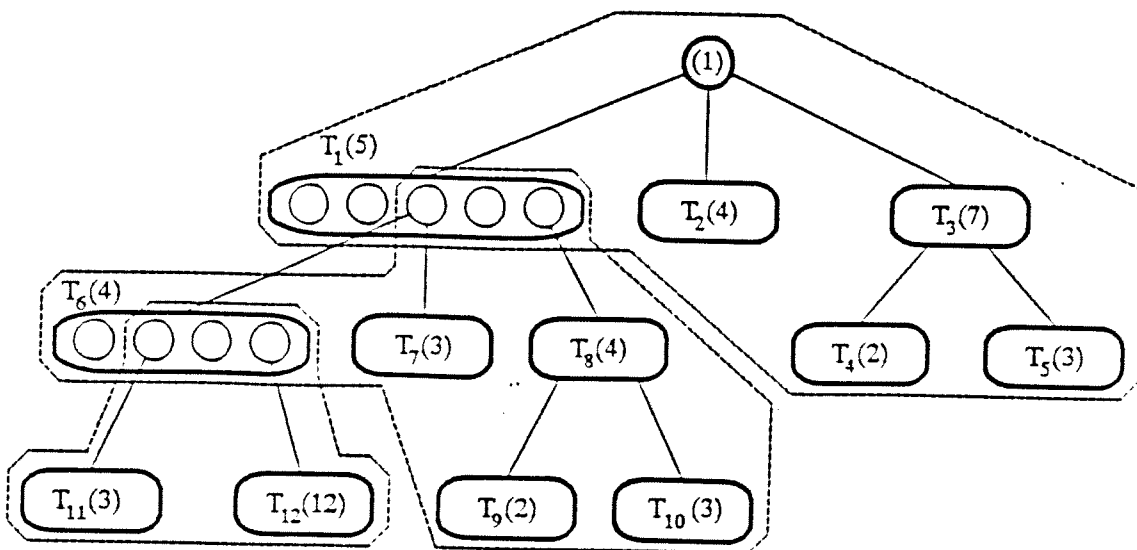
In this Section we will establish a one-to-one mapping between busy periods and a family of (labeled) planted planar trees, which provides a straightforward correspondence between deadline constraints and limited label sums of some subtrees. Due to this fact, we may relate the original problem of investigating the random variable  $SRD(T)$  to a counting problem regarding a special (sub)family  $\mathcal{B}_T$  of trees.

Consider the following diagram, which represents an example busy period.



According to our discrete time model, the horizontal axis is divided into equidistant cycles. Those cycles forming the busy period of interest are numbered consecutively; cycle 0 denotes the initial (idle) cycle. Task arrivals are shown by small lightnings with task-names above. The execution of a task is displayed by a horizontal line whose length equals the task execution time. The vertical level of a line, i.e., its vertical distance to the horizontal axis, represents the number of tasks not processed to completion at the beginning of that task. For the sake of readability, each such line is marked with the name of the corresponding task (and, sometimes, its task execution time).

There is an important relation between deadline constraints and the length of the so-called *sub busy periods*. A sub busy period is the epoch from the arrival of the first (new) task during the execution of a level 1 (or level 0) task to the end of the last cycle of the new task. For instance, looking at the cycle 0 in our example, one obtains the arrival of task  $T_1$ . Due to the nonpreemptive LCFS scheduling discipline, this task is badly off, because all tasks arriving before the beginning of the execution of  $T_1$  are preferred! Hence, if the length of a sub busy period is less or equal to  $T$ , all processed tasks are guaranteed to meet a service time deadline of  $T$  cycles. Conversely, if the length of a sub busy period is larger than  $T$ , at least the task having arrived first will miss its deadline.



To obtain the corresponding tree, a task is represented by an elliptical node which is labeled according to its task execution time, i.e., the label of a node is the number of cycles necessary for processing the task to completion. Equivalently, this labelling may be done by drawing the corresponding number of circles (each describing a cycle, of course) within the node.

The number of successors of a node equals the number of arrivals during the execution of the corresponding task. If a task has a task execution time of  $l$  cycles and will be scheduled to start at the  $i$ -th cycle ( $i \geq 0$ ) of the busy period, the execution will be completed at the end of the  $i + l - 1$ -th cycle since we are dealing with nonpreemptive LCFS scheduling. Thus, with  $t_i$ ,  $i \geq 0$  denoting the number of task arrivals during the  $i$ -th cycle of the busy period, the number of successors of the node is  $t_i + t_{i+1} + \dots + t_{i+l-1}$ . Successors are drawn from the left to the right, according to their arrival sequence.

Due to our construction, the outer leftmost (elliptical) nodes in the tree correspond to those tasks which both complete a sub busy period and start a new one, too. They are displayed in the equivalent labeling-style mentioned above. If such a node has no successors, it indicates the end of the whole busy period; at least one idle cycle follows.

Note, that the reconstruction of the busy period from a given tree is done by a right-to-left preorder traversal of all (elliptical) nodes of the tree.

Deadline constraints are reflected by suitable limits on the number of cycles. More precisely, the sum of the labels of nodes belonging to a sub busy period has to be less than the deadline  $T$ , for all sub busy periods, of course. In our example tree above, those nodes belonging to a specific sub busy period are fenced in by a dotted line.

Unfortunately, the fact that consecutive sub busy periods overlap one another, introduces unpleasant difficulties. Since two consecutive sub busy periods are pasted together at an outer leftmost node, (some of) its cycles have to be taken into account in both. On the other hand, to obtain the total number of cycles of a whole busy period, each cycle has to be counted exactly once. Hence, we are forced to investigate trees representing sub busy periods first, and paste them together in order to obtain whole busy periods.

### 3. COMBINATORICS

As mentioned in Section 1, a run denotes a sequence of busy periods not violating any task's deadline, followed by a busy period with at least one deadline violation. Let

$$b_{k,T} = \text{prob}\{\text{Length of a non violating busy period equals } k \text{ cycles}\}$$

and

$$B_T(u) = \sum_{k \geq 0} b_{k,T} u^k \quad (3.1)$$

be the corresponding PGF. The PGF of the random variable  $\text{SRD}(T)$ , i.e., the length of a successful run, is given by

$$S_T(u) = \sum_{k \geq 0} s_{k,T} u^k = \frac{1 - B_T(1)}{1 - B_T(u)}. \quad (3.2)$$

This follows from the fact that the PGF of the length of an arbitrary number of nonviolating busy periods is  $\sum_{n \geq 0} B_T(u)^n$ , and that the probability of the occurrence of the terminating violation busy period equals  $1 - B_T(1)$ .

As promised, we start our treatment concerning  $B_T(u)$  with studying the family  $\mathcal{B}_{i,j}$  of trees which correspond to sub busy periods starting with a label  $i$  node and completing with a label  $j$  node ( $i \geq 1, j \geq 1$ ). We shall use symbolic equations for the description of classes of combinatorial structures (i.e., families of trees), cf. [3] for an overview. To keep the notation simple, we defer attaching the necessary weights to the translation into generating functions.

We have the following decomposition:

$$\mathcal{B}_{i,j} = \mathcal{H}_j \mathcal{V}_{i-1} + \mathcal{E} \mathcal{H}_j \mathcal{V}_{i-2} + \mathcal{E}^2 \mathcal{H}_j \mathcal{V}_{i-3} + \cdots + \mathcal{E}^{i-2} \mathcal{H}_j \mathcal{V}_1 + \mathcal{E}^{i-1} \mathcal{H}_j. \quad (3.3)$$

The combinatorial objects used for building blocks have straightforward meaning.  $\mathcal{E}$  denotes a single cycle with no task arrivals,  $\mathcal{H}_j$  denotes a single cycle with at least one arrival, leading to the leftmost label  $j$  node.  $\mathcal{V}_k$  denotes a sequence of  $k \geq 1$  consecutive cycles with an arbitrary number of arrivals. To start with the most important one, we have the following symbolic equation:

$$\mathcal{V}_k = \textcircled{k} + \begin{array}{c} \textcircled{k} \\ | \\ \mathcal{V}_* \end{array} + \begin{array}{c} \textcircled{k} \\ / \quad \backslash \\ \mathcal{V}_* \quad \mathcal{V}_* \end{array} + \cdots + \begin{array}{c} \textcircled{k} \\ / \quad \backslash \\ \mathcal{V}_* \cdots \mathcal{V}_* \end{array} + \cdots$$

with  $\mathcal{V}_* = \sum_k \mathcal{V}_k$ . In order to translate the symbolic equation into an appropriate ordinary generating function (OGF), we have to attach suitable sizes and weights to each combinatorial object. If we attach sizes by ‘multiplying’ each elementary object (i.e., a node with label  $l$ ) by  $z^l$ , the size of a (composed) object (i.e., a tree) is the sum of its labels. Additionally, providing suitable probability weights leads to an equivalent of the OGF of a class of combinatorial structures (i.e., a family of trees), namely the PGF of the random variable to which it corresponds.

For example, recalling definition (1.2), the OGF of  $\mathcal{V}_*$  reads

$$V_*(z) = \sum_{k \geq 1} l_k V_k(z).$$

Due to the definition of the PGF of task arrivals within a cycle, we have

$$q_{n,k} = \text{prob}\{n \text{ task arrivals during } k \text{ (consecutive) cycles}\} = [z^n] A(z)^k.$$

for  $n \geq 0, k \geq 1$ . Obviously,  $[z^n] f(z)$  denotes the coefficient of  $z^n$  in the power series expansion of  $f(z)$ . Thus, the OGF of  $\mathcal{V}_k$  reads

$$\begin{aligned} V_k(z) &= q_{0,k} z^k + q_{1,k} z^k V_*(z) + \cdots + q_{n,k} z^k V_*(z)^n + \cdots \\ &= z^k \sum_{n \geq 0} q_{n,k} V_*(z)^n. \end{aligned}$$

Introducing the bivariate generating function

$$G(z, u) = \sum_{k \geq 0} l_k V_k(z) u^k,$$

one obtains  $V_*(z) = G(z, 1)$ . Multiplying the above equation for  $V_k(z)$  by  $l_k u^k$  and summing up for  $k \geq 1$  yields

$$\begin{aligned} G(z, u) &= \sum_{k \geq 1} l_k (zu)^k \sum_{n \geq 0} q_{n,k} G(z, 1)^n \\ &= \sum_{n \geq 0} G(z, 1)^n [w^n] \sum_{k \geq 1} l_k (A(w) zu)^k \\ &= L(zu A(G(z, 1))). \end{aligned}$$

Because of

$$V_1(z) = [l_1 u^1] G(z, u) = z A(G(z, 1)),$$

we find

$$\sum_{k \geq 1} l_k V_k(z) u^k = G(z, u) = L(u V_1(z)) = \sum_{k \geq 1} l_k V_1(z)^k u^k,$$

hence  $V_k(z) = V_1(z)^k$  and  $V_*(z) = G(z, 1) = L(V_1(z))$ . Substituting the latter in the equation for  $V_1(z)$  above, we obtain

$$V_1(z) = z A(L(V_1(z))) = z P(V_1(z)).$$

As we might have expected, this is the generating function of a family of simply generated trees, cf. [6]. This function appeared in our investigations concerning preemptive LCFS scheduling ( $B(z)$ , see [8]), too.

At next, we look at  $\mathcal{H}_j$ ,  $j \geq 1$ . The symbolic equation reads

$$\mathcal{H}_j = \begin{array}{c} \textcircled{1} \\ | \\ T_j \end{array} + \begin{array}{c} \textcircled{1} \\ / \quad \backslash \\ T_j \quad \mathcal{V}_* \end{array} + \dots + \begin{array}{c} \textcircled{1} \\ / \quad \backslash \quad \backslash \\ T_j \quad \mathcal{V}_* \quad \dots \quad \mathcal{V}_* \end{array} + \dots$$

with  $T_j$  denoting a label  $j$  node. Obviously, the corresponding OGF is  $T_j(z) = l_j z^j$ .

Since each combinatorial object in  $\mathcal{H}_j$  corresponds to an object in  $\mathcal{V}_1$ , where the leftmost successor  $\mathcal{V}_*$  (at the top level) is replaced by  $T_j$ , we may omit the detailed translation of the symbolic equation and write down the result immediately:

$$H_j(z) = l_j z^j \frac{V_1(z) - a_0 z}{L(V_1(z))}.$$

Note, that the term  $a_0 z$  corresponds to the ‘smallest’ tree in  $\mathcal{V}_1$ , which consists of the root only (no arrivals during the corresponding cycle).

The OGF for  $\mathcal{E}$  is straightforward; mentioning definition (1.2), we have

$$E(z) = a_0 z.$$

We summarize the results in the following lemma.

LEMMA 3.1. With the notations above, the ordinary generating functions of  $\mathcal{V}_k$ ,  $\mathcal{H}_j$  and  $\mathcal{E}$ , respectively, are given by

$$\begin{aligned} V_k(z) &= B(z)^k \quad \text{for } k \geq 1, \\ H_j(z) &= l_j z^j \frac{B(z) - a_0 z}{L(B(z))} \quad \text{for } j \geq 1, \\ E(z) &= a_0 z, \end{aligned}$$

where  $B(z)$  denotes the solution of

$$B(z) = zA(L(B(z))) = zP(B(z)).$$

Now we are able to translate symbolic equation (3.3) into the appropriate PGF. For reasons which will become more clear when pasting sub busy periods together, we shall attach two different sizes to a structure from that class. Roughly speaking, the size represented by  $z$  is responsible for counting the length of the corresponding sub busy period w.r.t. deadline properties. A different size is represented by the variable  $u$ . It counts the contributions of the corresponding sub busy period to the overall length of the whole busy period; remember our remarks at the end of Section 2. We find

$$B_{i,j}(z, u) = \sum_{l=0}^{i-1} E(1)^{i-1-l} H_j(zu) B(zu)^l u^{-1-l}.$$

Note, that we should have no contributions from  $\mathcal{E}$ , both for deadline counting and the overall size, thus  $E(1)$  is used. The last term  $u^{-1-l}$  makes the difference in the size counted by  $z$  and  $u$ . The  $l+1$  cycles within the initial label  $i$  node, i.e., the 'roots' of  $\mathcal{H}_j$  and  $\mathcal{V}_1^l$ , must be counted in  $z$  only (deadlines), not in  $u$ . The latter is done in the preceding sub busy period!

That is, for a sub busy period starting with a label  $i$  and terminating with a label  $j$  node,  $[z^t][u^n]B_{i,j}(z, u)$  is the probability that all tasks meet a deadline of  $t$  cycles (and no smaller one), contributing  $n$  cycles to the length of the whole busy period.

Simplifying the expression above yields

$$\begin{aligned} B_{i,j}(z, u) &= \frac{B(zu) - a_0 zu}{L(B(zu))} l_j (zu)^j \sum_{l=0}^{i-1} a_0^{i-1-l} B(zu)^l u^{-1-l} \\ &= \left( \left( \frac{B(zu)}{u} \right)^i - a_0^i \right) \frac{B(zu) - a_0 zu}{L(B(zu))} \cdot \frac{1}{B(zu) - ua_0} l_j (zu)^j. \end{aligned} \quad (3.4)$$

Now, we shall try to paste sub busy periods together. In order to enable deadline counting in each sub busy period, we are forced to use *different counting variables*  $z_k$  instead of  $z$ . Let  $\mathcal{B}_{i,j}^k$  denote the family of trees, which are formed by pasting together exactly  $k \geq 1$  sub busy periods. For example, we have

$$\mathcal{B}_{i,j}^2 = \sum_{k \geq 1} \mathcal{B}_{i,k} \mathcal{B}_{k,j},$$



the corresponding (multivariate) generating function reads

$$B_{i,j}^2(z_2, z_1; u) = \sum_{k \geq 1} B_{i,k}(z_2, u) B_{k,j}(z_1, u).$$

To keep notations simple, we introduce the following abbreviations:

$$B_{i,j}(z, u) = S_i(z, u) I(z, u) T_j(z, u),$$

cf. equation (3.4). Using this, we obtain

$$\begin{aligned} B_{i,j}^2(z_2, z_1; u) &= S_i(z_2, u) I(z_2, u) \sum_{k \geq 1} T_k(z_2, u) S_k(z_1, u) I(z_1, u) T_j(z_1, u) \\ &= S_i(z_2, u) I(z_2, u) \sum_{k \geq 1} l_k(z_2 u)^k \left( \left( \frac{B(z_1 u)}{u} \right)^k - a_0^k \right) I(z_1, u) T_j(z_1, u) \\ &= S_i(z_2, u) I(z_2, u) [L(z_2 B(z_1 u)) - L(a_0 z_2 u)] I(z_1, u) T_j(z_1, u). \end{aligned}$$

Note, that overlapping of sub busy periods is reflected by the ‘connecting function’ within the brackets. The ‘starting’ and ‘trailing’ functions  $S_i(., u)$  and  $T_j(., u)$  appear in the expression again; thus we may use this technique repeatedly to construct the general term:

$$\begin{aligned} B_{i,j}^k(z_k, \dots, z_1; u) &= S_i(z_k, u) I(z_k, u) \cdot \\ &\quad (L(z_k B(z_{k-1} u)) - L(a_0 z_k u)) I(z_{k-1}, u) \cdot \\ &\quad (L(z_{k-1} B(z_{k-2} u)) - L(a_0 z_{k-1} u)) I(z_{k-2}, u) \cdot \\ &\quad \vdots \\ &\quad (L(z_2 B(z_1 u)) - L(a_0 z_2 u)) I(z_1, u) \cdot \\ &\quad T_j(z_1, u) \end{aligned}$$

To construct a whole busy period consisting of exactly  $k$  sub busy periods, we have to deal with the decomposition

$$C^k = \mathcal{U} \sum_{j \geq 1} B_{1,j}^k \mathcal{E}_j.$$

$\mathcal{U}$  denotes a single cycle forming the initial cycle of the first sub busy period, its OGF is  $U(z) = z$ .  $\mathcal{E}_j$  is a label  $j$  node with no arrivals: according to Lemma 3.1, we obtain the OGF  $E_j(z) = E(z)^j = (a_0 z)^j$ . Translating the symbolic equation above, we find

$$C^k(z_k, \dots, z_1; u) = u \sum_{j \geq 1} B_{1,j}^k(z_k, \dots, z_1; u) E(1)^j.$$

Note, that we do not count cycles resulting from the terminating idle period, i.e.,  $\mathcal{E}_j$ . We easily obtain

$$\begin{aligned}
C^k(z_k, \dots, z_1; u) &= u S_1(z_k, u) I(z_k, u) \cdot \\
&\quad (L(z_k B(z_{k-1} u)) - L(a_0 z_k u)) I(z_{k-1}, u) \cdot \\
&\quad \vdots \\
&\quad (L(z_2 B(z_1 u)) - L(a_0 z_2 u)) I(z_1, u) \cdot \\
&\quad L(a_0 z_1 u) \\
&= u \left( \frac{B(z_k u)}{u} - a_0 \right) \frac{B(z_k u) - a_0 z_k u}{L(B(z_k u))} \cdot \frac{1}{B(z_k u) - u a_0} \cdot \\
&\quad (L(z_k B(z_{k-1} u)) - L(a_0 z_k u)) \frac{B(z_{k-1} u) - a_0 z_{k-1} u}{L(B(z_{k-1} u))} \cdot \frac{1}{B(z_{k-1} u) - u a_0} \cdot \\
&\quad \vdots \\
&\quad (L(z_2 B(z_1 u)) - L(a_0 z_2 u)) \frac{B(z_1 u) - a_0 z_1 u}{L(B(z_1 u))} \cdot \frac{1}{B(z_1 u) - u a_0} \cdot \\
&\quad L(a_0 z_1 u).
\end{aligned}$$

Obviously, a busy period with no sub busy periods, that is, an idle cycle, has the symbolic equation  $\mathcal{UE}_1$ . The corresponding OGF is very simple:

$$C^0(u) = a_0 u.$$

Since a whole busy period may consist of an arbitrary number of sub busy periods not exceeding  $T$  cycles (for deadline counting, of course), we are forced to study

$$B_T(u) = a_0 u + \sum_{k \geq 1} [z_k^T] \cdots [z_1^T] \frac{1}{1 - z_k} \cdot \frac{1}{1 - z_{k-1}} \cdots \frac{1}{1 - z_1} C^k(z_k, \dots, z_1; u),$$

which is the PGF of the length of an arbitrary busy period containing no deadline violation. Introducing the abbreviations  $y_k = z_k u$  and

$$\begin{aligned}
D^k(y_k, \dots, y_1; u) &= \frac{1}{1 - y_k/u} \cdot \frac{1}{1 - y_{k-1}/u} \cdots \frac{1}{1 - y_1/u} C^k(y_k/u, \dots, y_1/u; u) \\
&= \frac{1}{1 - y_k/u} \cdot \frac{B(y_k) - a_0 y_k}{L(B(y_k))} \cdot \\
&\quad \frac{1}{1 - y_{k-1}/u} \cdot \frac{L(\frac{y_k}{u} B(y_{k-1})) - L(a_0 y_k)}{B(y_{k-1}) - u a_0} \cdot \frac{B(y_{k-1}) - a_0 y_{k-1}}{L(B(y_{k-1}))} \cdot \\
&\quad \vdots \\
&\quad \frac{1}{1 - y_1/u} \cdot \frac{L(\frac{y_2}{u} B(y_1)) - L(a_0 y_2)}{B(y_1) - u a_0} \cdot \frac{B(y_1) - a_0 y_1}{L(B(y_1))} \cdot \\
&\quad L(a_0 y_1)
\end{aligned}$$

we find

$$B_T(u) = a_0 u + \sum_{k \geq 1} u^{kT} [y_k^T] \cdots [y_1^T] D^k(y_k, \dots, y_1; u). \quad (3.6)$$

#### 4. ASYMPTOTICS

Looking more closely at the (delicate) expression for  $B_T(u)$ , one obtains non-trivial interdependencies among the coefficients  $[y_k^T], \dots, [y_1^T]$ , resulting from the ‘connecting functions’  $L(\frac{y_i}{u} B(y_{i-1}))$ . Hence, a direct extraction of the desired coefficients yields terribly complicated expressions, at first (and possibly second) sight far away from tractability. Thus, we shall use the powerful tool of singularity analysis of generating functions instead; see [3] for a survey.

Such techniques are based on the fact that the asymptotic behaviour of a Taylor coefficient  $[z^n]f(z)$  of an analytic function largely depends on the behaviour of  $f(z)$  near its dominant singularities, i.e., the ones of smallest modulus. In fact, restricting ourselves to functions having only one singularity  $z = \zeta$  on their circle of convergence, is it possible to deal with local expansions of  $f(z)$  near  $\zeta$ . For instance, if  $f(z) = O(g(z))$  for  $z \rightarrow \zeta$ , we have  $[z^n]f(z) = O([z^n]g(z))$  for  $n \rightarrow \infty$ , under fairly general conditions; consider [4] for a rigorous treatment concerning suitable ‘scales’ of functions  $g(z)$  and appropriate *transfer lemmas*. An example is the following

**LEMMA 4.1 (TRANSFER LEMMA).** *Assume that, with the sole exception of the (real and positive) singularity  $z = \zeta > 0$ , the function  $f(z)$  is analytic in the indented disk  $\Delta_\zeta(\eta, \varphi) = \{z : |z| \leq \zeta + \eta, |\arg(z - \zeta)| \geq \varphi, z \neq \zeta\}$ , where  $\eta > 0$  and  $0 < \varphi < \pi/2$ . Assume further that as  $z$  tends to  $\zeta$  in  $\Delta_\zeta(\eta, \varphi)$*

$$f(z) = O((1 - z/\zeta)^\alpha)$$

*for some real number  $\alpha$ . Then the  $n$ -th Taylor coefficient of  $f(z)$  satisfies*

$$[z^n]f(z) = O(n^{-\alpha-1}\zeta^{-n}).$$

Note, that this lemma requires analytic continuation of  $f(z)$  beyond its circle of convergence, but only order of growth information (and no side-conditions à la Tauber or Darboux). The proof is based on estimations of Cauchy’s formula using a suitable contour in  $\Delta_\zeta(\eta, \varphi)$ , see [4] for details.

Using Lemma 4.1, we shall establish the (well-known, cf. [6]) asymptotic behaviour of the  $n$ -th Taylor coefficient of

$$B(z) = \sum_{n \geq 0} b_n z^n,$$

which denotes the (non-negative, that is,  $b_n \geq 0$ ) solution of the functional equation  $B(z) = zP(B(z))$  with  $P(w) = A(L(w))$ , cf. Lemma 3.1. The following conditions are to be met:

- (i)  $A(0) > 0$ , cf. Section 2.
- (ii)  $L(0) = 0$ , cf. Section 2.

(iii)  $P''(1) > 0$ , which implies that for some  $l \geq 2$

$$P(w) = \sum_{n \geq 0} p_n w^n = p_0 + p_1 w + p_l w^l + O(w^{l+1}) \quad \text{for } w \rightarrow 0$$

where  $p_0 = a_0 > 0$  and  $p_l > 0$ .

- (iv) The equation  $P(w) - wP'(w) = 0$  has a real solution  $w = \tau > 1$ . Note, that this forces  $P'(1) < 1$ , as can be shown by simple geometric arguments.
- (v)  $P(w)$  has a radius of convergence larger than  $\tau$ .
- (vi)  $L(w)$  has a radius of convergence larger than  $\tau^2/P(\tau)$ . This condition will become meaningful later in this section.

Providing this, we are able to state the following

LEMMA 4.2 (EXPANSION OF  $B(z)$ ). *With the notations and conditions above, the function  $B(z)$  is analytic in a domain  $\Delta_\rho(\eta, \varphi)$  with some  $\eta > 0$  and  $0 < \varphi < \pi/2$ . There is only one algebraic singularity  $z = \rho$  on its circle of convergence, in whose neighborhood*

$$B(z) = \tau - b \cdot (1 - z/\rho)^{1/2} + O(1 - z/\rho) \quad \text{for } z \rightarrow \rho, z \in \Delta_\rho(\eta, \varphi)$$

with  $b = \sqrt{\frac{2P(\tau)}{P''(\tau)}}$  holds. The asymptotic expansion of  $b_n = [z^n]B(z)$  reads

$$b_n = \frac{b}{2\sqrt{\pi}} n^{-3/2} \rho^{-n} + O(n^{-2} \rho^{-n}) \quad \text{for } n \rightarrow \infty.$$

**Proof:** Let

$$F(z, w) = \sum_{i+j \geq 1} f_{ij} z^i w^j = zP(w).$$

The following facts are easily established:

- (1) All  $f_{ij}$  (and  $b_n$ , of course) are non-negative and condition (iii) implies  $f_{ij} > 0$  for some  $j \geq 2$ . In addition,  $f_{00} > 0$  by condition (i) and  $f_{01} = 0 \neq 1$ .
- (2) Let

$$\rho = \frac{\tau}{P(\tau)},$$

we have  $1 < \rho < \tau$ . The lower bound follows from the fact that  $f(x) = x/P(x)$  is strictly monotonic for  $x < \tau$  and  $f(1) = 1$ , the upper bound is obvious since  $P(\tau) > 1$ . It is clear from condition (v) that the point  $(\rho, \tau)$  lies within the region where  $F(z, w)$  converges (absolutely) and that

$$\begin{aligned} F(\rho, \tau) &= \tau \\ F_w(\rho, \tau) &= 1, \end{aligned}$$

remember condition (iv).

- (3) There is some  $j > i \geq 1$  with  $\gcd(i, j) = 1$  such that  $b_i b_j > 0$ . Provided that  $B(z)$  exists, we have  $b_0 = B(0) = 0 \cdot P(B(0)) = 0$ , hence  $B(z) = O(z)$  for  $z \rightarrow 0$ . Bootstrapping yields

$$B(z) = zP(O(z)) = zp_0 + O(z^{m+1}) \quad \text{for } z \rightarrow 0$$

with  $m \geq 1$  by virtue of condition (iii). A second step establishes

$$B(z) = zP(zp_0 + O(z^{m+1})) = zp_0 + p_m p_0^m z^{m+1} + O(z^{m+2}) \quad \text{for } z \rightarrow 0.$$

Hence,  $i = 1$  and  $j = m + 1 \geq 2$  provide the required result.

At this point, all conditions necessary for the application of Theorem 2 of [7] are established. It provides the conclusion that  $B(z)$  is analytic, has an algebraic singularity  $z = \rho$  on its circle of convergence (but no others) and  $B(\rho) = \tau$ . The required expansion for  $z \rightarrow \rho$  follows from Theorem 5 of [1]. Since  $B(z)$  may be continued analytically beyond its circle of convergence for all  $z = \zeta$  with  $|\zeta| = \rho$  but  $\zeta \neq \rho$ , as can be shown by using the Implicit Function Theorem,  $B(z)$  is analytic in a domain  $\Delta_\rho(\eta, \varphi)$  with  $\eta > 0$  and  $0 < \varphi < \pi/2$ . The same is true for  $g(z) = -(1 - z/\rho)^{1/2}$ , whose  $n$ -th Taylor coefficient  $g_n$  has the well-known expansion

$$g_n = \frac{1}{2\sqrt{\pi}} n^{-3/2} \rho^{-n} + O(n^{-5/2} \rho^{-n}) \quad \text{for } n \rightarrow \infty.$$

Thus, we are permitted to apply Lemma 4.1 to  $B(z) - \tau + b \cdot (1 - z/\rho)^{1/2}$ , which establishes the asymptotic expansion of  $b_n$  as asserted. ■

Keep in mind that the remainder  $O(1 - z/\rho)$  in the expansion of  $B(z)$  represents a function which is analytic in  $\Delta_\rho(\eta, \varphi)$ ! By the way, it is easy to improve this remainder term to  $O((1 - z/\rho)^{3/2})$  by using the more accurate expansion  $B(z) = \tau - b \cdot (1 - z/\rho)^{1/2} + c \cdot (1 - z/\rho) + O((1 - z/\rho)^{3/2})$  for  $z \rightarrow \rho$ ,  $z \in \Delta_\rho(\eta, \varphi)$ . The linear term contributes nothing to the  $n$ -th Taylor coefficient, hence

$$b_n = \frac{b}{2\sqrt{\pi}} n^{-3/2} \rho^{-n} + O(n^{-5/2} \rho^{-n}) \quad \text{for } n \rightarrow \infty. \quad (4.1)$$

After these preliminary discussion we shall give a short and informal overview how to proceed with the investigation of  $B_T(u)$ . Since we are interested in (factorial) moments of  $\text{SRD}(T)$ , that is, derivatives of  $S_T(u)$  at  $u = 1$  (cf. equation (3.2)), we have to deal with derivatives of  $B_T(u)$  evaluated at  $u = 1$ . Thus, in what follows we assume  $u$  to be a complex parameter in the closed disk  $D(1, \nu) = \{z : |z - 1| \leq \nu\}$  for some arbitrary small  $\nu > 0$ .

Looking more closely at  $y_1$ -related terms in  $B_T(u)$ , that is

$$\frac{1}{1 - y_1/u} \cdot \frac{L(\frac{y_2}{u} B(y_1)) - L(a_0 y_2)}{B(y_1) - u a_0} \cdot \frac{B(y_1) - a_0 y_1}{L(B(y_1))} \cdot L(a_0 y_1) \quad (4.2)$$

our task is the determination of the  $T$ -th Taylor coefficient  $[y_1^T]$  in this multivariate function, which is analytic for  $y_1, y_2$  in a neighborhood of 0 and  $u \in D(1, \nu)$ . Due to general Theorems (Cauchy's formula for multivariate analytic functions),  $[y_1^T]f(y_2, y_1, u)$  is an analytic function of  $y_2$  and  $u$ , too. In addition, it is not hard to prove that the statement of Lemma 4.1 remains valid for a multivariate analytic function. For example, if

$$f(z, w) = O(g(w)(1 - z/\zeta)^\alpha) \quad \text{for } z \rightarrow \rho,$$

uniformly w.r.t.  $w$ , it follows that

$$[z^n]f(z, w) = O(g(w)n^{-1-\alpha}\zeta^{-n}) \quad \text{for } n \rightarrow \infty,$$

uniformly in  $w$ , too. Again, keep in mind that the latter  $O(\cdot)$  represents a function which is analytic in  $w$ !

Returning to our original function, we obtain three 'sources' of singularities,

- (1) a (removeable) simple pole at  $y_1 = \zeta(u) < 1$ , resulting from  $B(\zeta(u)) = a_0 u$ ,
- (2) a simple pole at  $y_1 = u$ ,
- (3) an algebraic singularity at  $y_1 = \rho$  resulting from functions involving  $B(y_1)$ .

The fact, that  $y_1 = \zeta(u)$  is a removeable singularity, i.e., that there is no singularity at all, is easily established by taking into account the zero of  $L(\frac{y_2}{u}B(y_1)) - L(a_0 y_2)$  at  $y_1 = \zeta(u)$ .

Remembering  $\rho > 1$  it follows that  $y_1 = u$  is the singularity with the smallest modulus; in fact we choose  $\nu$  small enough, i.e.,  $1 + \nu < \rho$ . The appropriate contribution to  $[y_1^T]$  is easily determined via *subtracted singularities*:

$$\frac{L(\frac{y_2}{u}B(u)) - L(a_0 y_2)}{L(B(u))} L(a_0 u) \cdot u^{-T}.$$

Investigating the behaviour of (4.2) near the 'next' singularity  $y_1 = \rho$  it turns out that  $B(y_1) - a_0 y_1$  and  $L(B(y_1))$  obey expansions similar to  $B(y_1)$ . The function  $L(a_0 y_1)$  has a radius of convergence larger than  $\rho$  by virtue of condition (vi), i.e., is well-behaved in a neighborhood of  $y_1 = \rho$ . Hence, the only remaining difficulty concerns the term containing the 'connecting function', i.e.,

$$\frac{L(\frac{y_2}{u}B(y_1)) - L(a_0 y_2)}{B(y_1) - u a_0}.$$

But, using the mentioned extension of our devices it is possible to attack this multivariate analytic function, too. Since  $y_1$  comes up with  $B(y_1)$ , one feels that  $L(\frac{y_2}{u}B(y_1))$  should have an algebraic singularity at  $y_1 = \rho$ , independent of  $y_2$ ! Due to the fact that, at our next 'stage',  $y_2$  will play the role of  $y_1$  and  $y_3$  the one of  $y_2$ , it is obvious to ask for the behaviour in a neighborhood of  $y_2 = \rho$  (and  $y_2 = u$ , resulting from the subtracted singularity term for  $y_2$ , too). However, since  $y_2$  appears in conjunction with the well-behaved function  $L(\cdot)$  only, we may expect inferior influences here: To make a long story short, we assert that it is possible to determine a uniform expansion

$$\frac{L(\frac{y_2}{u}B(y_1)) - L(a_0 y_2)}{B(y_1) - u a_0} = b(y_2, u) + c(y_2, u)(1 - y_1/\rho)^{1/2} + O(1 - y_1/\rho) \quad \text{for } y_1 \rightarrow \rho$$

where  $b(y_2, u)$  and  $c(y_2, u)$  denote well-behaved analytic functions of both  $y_2$  and  $u$ . The remainder  $O(1 - y_1/\rho)$  represents a multivariate analytic function, too, and the implied constant is independent of  $y_1, y_2$  and  $u$ .

Note, that although it is impossible to separate the ‘connecting function’ directly, i.e., to split up  $L(\frac{y_2}{u}B(y_1))$  into a product  $f(y_1)g(y_2)$ , an *asymptotic separation* succeeded!

Putting all terms together, we obtain a uniform expansion for (4.2) at  $y_1 = \rho$ , similar to the expansion above:

$$\beta(y_2, u)L(a_0\rho) + \gamma(y_2, u)L(a_0\rho)(1 - y_1/\rho)^{1/2} + O(1 - y_1/\rho) \quad \text{for } y_1 \rightarrow \rho.$$

Note, that the terms  $L(a_0\rho)$  represent the contribution resulting from the terminating function  $L(a_0y_1)$ .

The subtracted term resulting from the simple pole  $y_1 = u$  is meaningless for the analysis of the singularity  $y_1 = \rho$  since  $(1 - y_1/u)^{-1}$  is analytic for all  $y_1 \neq u$ . Using transfer lemmas, the desired coefficient  $[y_1^T]$  finally yields

$$\alpha(y_2, u)L(a_0u)u^{-T} - \frac{1}{2\sqrt{\pi}}\gamma(y_2, u)L(a_0\rho)T^{-3/2}\rho^{-T} + O(T^{-2}\rho^{-T}) \quad \text{for } T \rightarrow \infty$$

with both  $\alpha(y_2, u)$  and  $\gamma(y_2, u)$  analytic at  $y_2 = \rho$ ; the ‘elimination’ of  $y_1$  is complete.

Now, the same procedure may be used for the extraction of  $[y_2^T]$  (hence, for all  $[y_i^T]$ ) since the related terms are almost the same. In fact, the only difference springs from replacing  $L(a_0y_1)$  by  $\alpha(y_2, u)$  and  $\gamma(y_2, u)$ , respectively! Using this simple iterative scheme (leading to a recurrence relation) it is possible to compute an asymptotic expansion

$$B_T(u) = B(u) - R(u)u^T T^{-3/2}\rho^{-T} + O(u^T T^{-2}\rho^{-T})$$

uniformly valid for  $u \in D(1, \nu)$ . By virtue of a general theorem concerning uniform expansions we may differentiate this expansion in order to derive  $B_T^{(m)}(1)$  for an arbitrary but fixed  $m$ .

We start our detailed treatment with providing some ‘building blocks’, that is, asymptotic expansions of the functions involved. At first we look at  $C(z) = L(B(z))$  which obviously denotes the (positive) solution of  $C(z) = L(zA(C(z)))$ . Thus, the same procedure as in the proof of Lemma 4.2 might be used. This would establish that  $C(z)$  has exactly one algebraic singularity  $z = \rho$  on its circle of convergence and  $C(\rho) = L(\tau)$ , i.e., provide an asymptotic expansion similar to  $B(z)$ .

But, since we need the uniform asymptotic expansion of  $C(z, w) = L(wB(z))$  for an arbitrary complex value  $w \in \Delta_\rho(\eta, \varphi)$ , too (which covers  $C(z)$  above), we shall use another idea. Note however, that  $C(z, w)$  is a solution of  $C(z, w) = L(wzP(L^{[-1]}(C(z, w))/w))$  which (formally) leads to an algebraic singularity  $z = \rho$  and  $C(\rho, w) = L(w\tau)$ .

Our alternative approach is based on condition (vi), which guarantees that  $v = w\tau$  lies within the radius of convergence of  $L(\cdot)$  for all  $w \in D(0, \rho + \varepsilon)$  for some  $\varepsilon > 0$  sufficiently small. Using the Taylor expansion at  $w\tau$ , i.e.,

$$L(v) = \sum_{n \geq 0} \frac{L^{(n)}(w\tau)}{n!} (v - w\tau)^n,$$

valid for  $v \in D(w\tau, \varepsilon)$ , and substituting

$$v = wB(z) = w\tau - wb(1 - z/\rho)^{1/2}(1 + r(z))$$

with  $r(z) = O(1 - z/\rho)^{1/2}$  (analytic for  $z \in \Delta_\rho(\eta, \varphi)$ ), we have for  $z$  sufficiently close to  $\rho$

$$\begin{aligned} L(wB(z)) &= \sum_{n \geq 0} \frac{L^{(n)}(w\tau)}{n!} \left( -wb(1 - z/\rho)^{1/2}(1 + r(z)) \right)^n \\ &= L(w\tau) - bwL'(w\tau)(1 - z/\rho)^{1/2} + O(1 - z/\rho) \quad \text{for } z \rightarrow \rho, z \in \Delta_\rho(\eta, \varphi), \end{aligned}$$

where the  $O(\cdot)$ -term is uniformly valid for  $w \in D(0, \rho + \varepsilon)$  (and denotes an analytic function of both  $z$  and  $w$ , of course)! Using this result, we find for  $y_{k-1} \rightarrow \rho^\ddagger$

$$\begin{aligned} \frac{1}{L(B(y_{k-1}))} &= \frac{1}{L(\tau) - bL'(\tau)(1 - y_{k-1}/\rho)^{1/2}(1 + O((1 - y_{k-1}/\rho)^{1/2}))} \\ &= \frac{1}{L(\tau)} + \frac{bL'(\tau)}{L(\tau)^2}(1 - y_{k-1}/\rho)^{1/2} + O(1 - y_{k-1}/\rho) \end{aligned}$$

and

$$L\left(\frac{y_k}{u}B(y_{k-1})\right) = L\left(\frac{y_k}{u}\tau\right) - b\frac{y_k}{u}L'\left(\frac{y_k}{u}\tau\right)(1 - y_{k-1}/\rho)^{1/2} + O(1 - y_{k-1}/\rho)$$

uniformly for  $y_k \in D(0, \rho + \varepsilon)$  and  $u \in D(1, \nu)$ . The latter and Lemma 4.2 yield the asserted expansion for

$$Q(y_k, y_{k-1}, u) = \frac{L\left(\frac{y_k}{u}B(y_{k-1})\right) - L(a_0 y_k)}{B(y_{k-1}) - u a_0},$$

namely

$$\begin{aligned} Q(y_k, y_{k-1}, u) &= \frac{L\left(\frac{y_k}{u}\tau\right) - L(a_0 y_k) - b\frac{y_k}{u}L'\left(\frac{y_k}{u}\tau\right)(1 - y_{k-1}/\rho)^{1/2} + O(1 - y_{k-1}/\rho)}{\tau - a_0 u - b(1 - y_{k-1}/\rho)^{1/2} + O(1 - y_{k-1}/\rho)} \\ &= \frac{L\left(\frac{y_k}{u}\tau\right) - L(a_0 y_k)}{\tau - a_0 u} + \\ &\quad \left( \frac{b(L\left(\frac{y_k}{u}\tau\right) - L(a_0 y_k))}{(\tau - a_0 u)^2} - \frac{b\frac{y_k}{u}L'\left(\frac{y_k}{u}\tau\right)}{\tau - a_0 u} \right) (1 - y_{k-1}/\rho)^{1/2} + \\ &\quad O(1 - y_{k-1}/\rho) \quad \text{for } y_{k-1} \rightarrow \rho, \end{aligned}$$

uniformly valid for  $y_k \in D(0, \rho + \varepsilon)$  and  $u \in D(1, \nu)$ . Remember our remarks in the preceding overview, especially concerning the functions  $b(y_2, u)$  and  $c(y_2, u)$ !

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$\ddagger$ In what follows we use the notation  $y_{k-1} \rightarrow \rho$  as an abbreviation for  $y_{k-1} \rightarrow \rho$  where  $y_{k-1} \in \Delta_\rho(\eta, \varphi)$ .



For the sake of completeness we mention the trivial expansions

$$\frac{1}{1 - y_{k-1}/u} = \frac{1}{1 - \rho/u} + O(1 - y_{k-1}/\rho),$$

$$B(y_{k-1}) - a_0 y_{k-1} = \tau - a_0 \rho - b(1 - y_{k-1}/\rho)^{1/2} + O(1 - y_{k-1}/\rho)$$

as  $y_{k-1} \rightarrow \rho$ , and

$$L(a_0 y_k) = L(a_0 \rho) + O(1 - y_k/\rho),$$

$$L\left(\frac{y_k}{u}\tau\right) = L\left(\frac{\rho}{u}\tau\right) + O(1 - y_k/\rho)$$

as  $y_k$  tends to  $\rho$ , uniformly for  $u \in D(1, \nu)$ .

Thus, all 'building blocks' for the expansion of the expressions à la (4.2) are present. Let  $t_{k-1}(y, u, T)$  denote a function analytic for  $y \in D(0, \rho + \varepsilon)$  and  $u \in D(1, \nu)$  with

$$t_{k-1}(y, u, T) = t_{k-1}(\rho, u, T)(1 + O(1 - y/\rho)) \quad \text{for } y \rightarrow \rho \quad (4.3)$$

uniformly for  $u \in D(1, \nu)$ ,  $k \geq 1$  and  $T \rightarrow \infty$ . Starting with  $t_1(y, u, T) = L(a_0 y)$  we shall investigate  $F_k(y_k, y_{k-1}) = F_k(y_k, y_{k-1}, u, T)$  for  $k \geq 2$ , which reads

$$F_k(y_k, y_{k-1}) = \frac{1}{1 - y_{k-1}/u} \cdot \frac{L\left(\frac{y_k}{u}B(y_{k-1})\right) - L(a_0 y_k)}{B(y_{k-1}) - u a_0} \cdot \frac{B(y_{k-1}) - a_0 y_{k-1}}{L(B(y_{k-1}))} t_{k-1}(y_{k-1}, u, T)$$

$$= \beta(y_k, u) t_{k-1}(\rho, u, T) + \gamma(y_k, u) t_{k-1}(\rho, u, T)(1 - y_{k-1}/\rho)^{1/2} +$$

$$O(t_{k-1}(\rho, u, T)(1 - y_{k-1}/\rho)) \quad \text{as } y_{k-1} \rightarrow \rho,$$

uniformly for  $y_k \in D(0, \rho + \varepsilon)$ ,  $u \in D(1, \nu)$ ,  $k \geq 1$  and  $T \rightarrow \infty$ . Moreover, the functions  $\beta(y_k, u)$  and  $\gamma(y_k, u)$  are analytic and

$$\beta(y_k, u) = \frac{1}{1 - \rho/u} \cdot \frac{L\left(\frac{y_k}{u}\tau\right) - L(a_0 y_k)}{\tau - u a_0} \cdot \frac{\tau - a_0 \rho}{L(\tau)}$$

and

$$\gamma(y_k, u) = \frac{1}{1 - \rho/u} \left[ b \left( \frac{L\left(\frac{y_k}{u}\tau\right) - L(a_0 y_k)}{(\tau - u a_0)^2} - \frac{\frac{y_k}{u} L'\left(\frac{y_k}{u}\tau\right)}{\tau - a_0 u} \right) \frac{\tau - a_0 \rho}{L(\tau)} - \right.$$

$$\left. \frac{L\left(\frac{y_k}{u}\tau\right) - L(a_0 y_k)}{\tau - u a_0} \cdot \frac{b}{L(\tau)} + \frac{L\left(\frac{y_k}{u}\tau\right) - L(a_0 y_k)}{\tau - u a_0} (\tau - a_0 \rho) \frac{b L'(\tau)}{L(\tau)^2} \right]$$

$$= \frac{b}{1 - \rho/u} \cdot \frac{\tau - a_0 \rho}{\tau - a_0 u} \cdot \frac{1}{L(\tau)} \left[ \frac{L\left(\frac{y_k}{u}\tau\right) - L(a_0 y_k)}{\tau - u a_0} - \frac{y_k}{u} L'\left(\frac{y_k}{u}\tau\right) - \right.$$

$$\left. \frac{L\left(\frac{y_k}{u}\tau\right) - L(a_0 y_k)}{\tau - a_0 \rho} + \frac{L'(\tau)}{L(\tau)} (L\left(\frac{y_k}{u}\tau\right) - L(a_0 y_k)) \right]. \quad (4.4)$$

The contributions resulting from the simple pole at  $y_{k-1} = u$  are easily determined via subtracted singularities, as already mentioned. The subtracted function is

$$S_k(y_k, y_{k-1}, u, T) = \frac{1}{1 - y_{k-1}/u} \alpha(y_k, u) t_{k-1}(u, u, T),$$

where

$$\alpha(y_k, u) = \frac{L\left(\frac{y_k}{u} B(u)\right) - L(a_0 y_k)}{L(B(u))}. \quad (4.5)$$

Its  $T$ -th Taylor coefficient is an analytic function for  $y_k \in D(0, \rho + \varepsilon)$  and  $u \in D(1, \nu)$  and reads

$$[y_{k-1}^T] S_k(y_k, y_{k-1}, u, T) = \alpha(y_k, u) t_{k-1}(u, u, T) u^{-T}. \quad (4.6)$$

Obviously,  $S_k(y_k, y_{k-1}, u, T)$  is analytic at  $y_{k-1} = \rho$ , hence

$$S_k(y_k, y_{k-1}, u, T) = \frac{1}{1 - \rho/u} \alpha(y_k, u) t_{k-1}(u, u, T) + O(t_{k-1}(u, u, T)(1 - y_{k-1}/\rho))$$

for  $y_{k-1} \rightarrow \rho$ , uniformly for  $y_k \in D(0, \rho + \varepsilon)$ ,  $u \in D(1, \nu)$ ,  $k \geq 1$  and  $T \rightarrow \infty$ . Thus, the function  $G_k(y_k, y_{k-1}, u, T) = F_k(y_k, y_{k-1}, u, T) - S_k(y_k, y_{k-1}, u, T)$  has no singularity at  $y_{k-1} = u$  but the same singularity  $y_{k-1} = \rho$  as  $F_k(y_k, y_{k-1}, u, T)$ . For  $y_{k-1} \rightarrow \rho$  we obtain the expansion

$$\begin{aligned} G_k(y_k, y_{k-1}, u, T) &= \beta(y_k, u) t_{k-1}(\rho, u, T) - \frac{1}{1 - \rho/u} \alpha(y_k, u) t_{k-1}(u, u, T) + \\ &\quad \gamma(y_k, u) t_{k-1}(\rho, u, T) (1 - y_{k-1}/\rho)^{1/2} + \\ &\quad O(t_{k-1}(u, u, T)(1 - y_{k-1}/\rho)) + O(t_{k-1}(\rho, u, T)(1 - y_{k-1}/\rho)), \end{aligned}$$

uniformly valid for  $y_k \in D(0, \rho + \varepsilon)$ ,  $u \in D(1, \nu)$ ,  $k \geq 1$  and  $T \rightarrow \infty$ . This finally establishes the asymptotic expansion of  $[y_{k-1}^T] F_k(y_k, y_{k-1}, u, T)$ ; using our Transfer Lemma, we obtain (that is, define) for  $k \geq 2$

$$\begin{aligned} t_k(y_k, u, T) &= [y_{k-1}^T] F_k(y_k, y_{k-1}, u, T) \\ &= \alpha(y_k, u) t_{k-1}(u, u, T) u^{-T} - \frac{1}{2\sqrt{\pi}} \gamma(y_k, u) t_{k-1}(\rho, u, T) T^{-3/2} \rho^{-T} + \\ &\quad O(t_{k-1}(u, u, T) T^{-2} \rho^{-T}) + O(t_{k-1}(\rho, u, T) T^{-2} \rho^{-T}) \end{aligned} \quad (4.7)$$

as  $T$  tends to infinity. Obviously, the first term springs up from the subtracted singularity term, cf. (4.6). Keep in mind that the (uniform) remainder terms represent bivariate analytic functions, say  $R_k(y_k, u, T)$  and  $\bar{R}_k(y_k, u, T)$ , both analytic for  $y_k \in D(0, \rho + \varepsilon)$  and  $u \in D(1, \nu)$ . As already mentioned, we have

$$t_1(y_1, u, T) = L(a_0 y_1).$$

This iterative scheme defines a sequence of analytic functions which are consistent with equation (4.3). To show this, we shall provide suitable lemmas: prior to those technical

details we should establish the connection between  $B_T(u)$  and the above. This relation is straightforward, cf. equation (3.6):

$$B_T(u) = a_0 u + \sum_{k \geq 1} u^{kT} [y_k^T] \frac{1}{1 - y_k/u} \cdot \frac{B(y_k) - a_0 y_k}{L(B(y_k))} t_k(y_k, u, T).$$

Since  $t_k(y_k, u, T)$  is analytic for  $y_k \in D(0, \rho + \varepsilon)$ , the desired coefficient  $[y_k^T]$  is easily evaluated. Providing the expansion

$$\frac{B(y_k) - a_0 y_k}{L(B(y_k))} = \frac{\tau - a_0 \rho}{L(\tau)} + \left( \frac{bL'(\tau)(\tau - a_0 \rho)}{L(\tau)^2} - \frac{b}{L(\tau)} \right) (1 - y_k/\rho)^{1/2} + O(1 - y_k/\rho)$$

for  $y_k \rightarrow \rho$ , we easily obtain

$$B_T(u) = a_0 u + \frac{B(u) - a_0 u}{L(B(u))} u^{-T} \sum_{k \geq 1} t_k(u, u, T) u^{kT} - \frac{b}{2\sqrt{\pi}} \cdot \frac{(\tau - a_0 \rho)L'(\tau) - L(\tau)}{(1 - \rho/u)L(\tau)^2} T^{-3/2} \rho^{-T} (1 + O(T^{-1/2})) \sum_{k \geq 1} t_k(\rho, u, T) u^{kT} \quad (4.8)$$

Obviously, the first sum results from a subtracted singularity term for  $y_k = u$ .

That's why we are interested in infinite sums involving  $t_k$ ; in order to justify our manipulations, we provide the lemmas promised. Instead of investigating  $t_k$  directly, we look at  $h_{k-1}(y, u, T) = t_k(y, u, T) u^{(k-1)T}$  in a somewhat generalized manner:

LEMMA 4.3 (SOLUTION OF A RECURRENCE RELATION). *Consider*

$$h_n(y, u, T) = f(y, u) h_{n-1}(u, u, T) + g(y, u, T) \psi^T h_{n-1}(\rho, u, T) + r(y, u, T) \psi^T h_{n-1}(u, u, T)$$

for  $n \geq 1$ ,  $\rho > 1$  and  $T \rightarrow \infty$ . In addition,

$$h_0(y, u, T) = h(y).$$

$f(y, u) \not\equiv 0$ ,  $g(y, u, T)$ ,  $r(y, u, T)$  and  $h(y)$  denote (multivariate) analytic functions for  $y \in D(0, \rho + \varepsilon)$ ,  $1 + \varepsilon > \rho$  and  $u \in D(1, \nu)$ ,  $1 + \nu < \rho$ .  $\psi$  is an abbreviation for  $u/\rho < 1$ . If  $\max_{u \in D(1, \nu)} |f(u, u)| = r < 1$ ,  $g(y, u, T) = O(T^m)$  and  $r(y, u, T) = O(T^l)$  uniformly for some arbitrary but fixed real  $m, l$ , we have the uniform bound

$$|h_n(y, u, T)| \leq CR^n$$

for all  $n \geq 0$ ,  $y \in D(0, \rho + \varepsilon)$ ,  $r < R < 1$  and  $T$  sufficiently large.

**Proof:** (By simultaneous induction). Let  $r < R < 1$  arbitrary,  $C_1 = \max_{u \in D(1, \nu)} |h(u)|$  and  $C = \max_{u \in D(1, \nu), y \in D(0, \rho + \varepsilon)} \{|h(y)|, C_1 |f(y, u)|/r\}$ . The implied constants of the  $O(\cdot)$ -terms concerning  $g(y, u, T)$  and  $r(y, u, T)$  are denoted by  $M$  and  $L$ , respectively. We shall show that

$$|h_n(u, u, T)| \leq C_1 R^n,$$

$$|h_n(y, u, T)| \leq CR^n.$$

The case  $n = 0$  is trivial; investigating the case  $n > 0$  yields

$$\begin{aligned} |h_n(u, u, T)| &\leq rC_1R^{n-1} + MT^m|\psi|^TCR^{n-1} + LT^l|\psi|^TC_1R^{n-1} \\ &= C_1rR^{n-1} + C_1\left(\frac{CMT^m}{C_1} + LT^l\right)|\psi|^TR^{n-1} \\ &\leq C_1R^n \end{aligned}$$

since  $(\frac{CMT^m}{C_1} + LT^l)|\psi|^T$  can be made less than  $R - r > 0$  provided that  $T$  is sufficiently large. Similarly,

$$\begin{aligned} |h_n(y, u, T)| &\leq |f(y, u)|C_1R^{n-1} + MT^m|\psi|^TCR^{n-1} + LT^l|\psi|^TC_1R^{n-1} \\ &\leq CrR^{n-1} + C(MT^m + \frac{C_1LT^l}{C})|\psi|^TR^{n-1} \\ &\leq CR^n, \end{aligned}$$

since  $(MT^m + \frac{C_1LT^l}{C})|\psi|^T \leq R - r$  for  $T$  sufficiently large. ■

The following lemma justifies equation (4.3):

**LEMMA 4.4 (EXPANSION OF THE SOLUTION OF A RECURRENCE RELATION).** *With the notations and conditions of Lemma 4.3 and the further suppositions  $f(u, u) \neq 0$ ,  $f(\rho, u) \neq 0$  and  $h(u) \neq 0$  for  $u \in D(1, \nu)$ , we obtain*

$$h_n(y, u, T) = h_n(\rho, u, T)(1 + O(1 - y/\rho)) \quad \text{for } y \rightarrow \rho$$

uniformly for  $u \in D(1, \nu)$ ,  $n \geq 0$  and  $T \rightarrow \infty$ .

**Proof:** By virtue of the Taylor expansion at  $y = \rho$  we may write  $f(y, u) = f(\rho, u) + f^+(y, u)(1 - y/\rho)$  (and similar for  $g$  and  $r$ ) and obtain

$$\begin{aligned} \frac{h_n(y, u, T)}{h_n(\rho, u, T)} &= 1 + \frac{f^+(y, u) + g^+(y, u, T)\psi^T \frac{h_{n-1}(\rho, u, T)}{h_{n-1}(u, u, T)} + r^+(y, u, T)\psi^T}{f(\rho, u) + g(\rho, u, T)\psi^T \frac{h_{n-1}(\rho, u, T)}{h_{n-1}(u, u, T)} + r(\rho, u, T)\psi^T}(1 - y/\rho) \\ &= 1 + \frac{f^+(y, u) + O(\psi^T)}{f(\rho, u) + O(\psi^T)}(1 - y/\rho) = 1 + O(1 - y/\rho) \quad \text{for } y \rightarrow \rho, \end{aligned}$$

uniformly for  $u \in D(1, \nu)$  and  $T \rightarrow \infty$ . To justify the last step we show (by induction) that for all  $n \geq 0$

$$\frac{h_n(\rho, u, T)}{h_n(u, u, T)} = O(1)$$

uniformly for  $u \in D(1, \nu)$  and  $T \rightarrow \infty$ . The case  $n = 0$  is trivial; for  $n > 0$  we find

$$\begin{aligned} \frac{h_n(\rho, u, T)}{h_n(u, u, T)} &= \frac{f(\rho, u) + g(\rho, u, T)\psi^T \frac{h_{n-1}(\rho, u, T)}{h_{n-1}(u, u, T)} + r(\rho, u, T)\psi^T}{f(u, u) + g(u, u, T)\psi^T \frac{h_{n-1}(\rho, u, T)}{h_{n-1}(u, u, T)} + r(u, u, T)\psi^T} \\ &= \frac{f(\rho, u) + O(\psi^T)}{f(u, u) + O(\psi^T)} = O(1). \end{aligned}$$

This completes the proof of Lemma 4.4. ■

For  $y = u$  and  $y = \rho$  let

$$H_y(u, T) = \sum_{n \geq 0} h_n(y, u, T).$$

Both infinite sums of analytic functions obey a uniform bound  $K \sum_{n \geq 0} R^n = K/(1 - R)$  according to Lemma 4.3. Hence, each one represents the limit of a uniformly convergent sequence of analytic functions for  $u \in D(1, \nu)$ , i.e., is an analytic function itself by virtue of the Theorem of Weierstraß. They are easily evaluated by summing up the recurrence relation for  $n \geq 1$ :

$$\begin{aligned} H_y(u, T) - h(y) &= f(y, u)H_u(u, T) + g(y, u, T)\psi^T H_\rho(u, T) + r(y, u, T)\psi^T H_u(u, T) \\ &= v(y, u, T)H_u(u, T) + w(y, u, T)H_\rho(u, T) \end{aligned}$$

where

$$\begin{aligned} v(y) &= v(y, u, T) = f(y, u) + r(y, u, T)\psi^T, \\ w(y) &= w(y, u, T) = g(y, u, T)\psi^T. \end{aligned}$$

We obtain

$$\begin{aligned} H_u &= H_u(u, T) = \frac{w(u)H_\rho(u, T) + h(u)}{1 - v(u)}, \\ H_\rho &= H_\rho(u, T) = \frac{v(\rho)H_u(u, T) + h(\rho)}{1 - w(\rho)} \end{aligned}$$

which yields

$$(1 - v(u))H_u = \frac{w(u)v(\rho)}{1 - w(\rho)}H_u + \frac{w(u)h(\rho)}{1 - w(\rho)} + h(u)$$

and finally

$$\begin{aligned} H_u &= \frac{w(u)h(\rho) + h(u)(1 - w(\rho))}{(1 - v(u))(1 - w(\rho)) - w(u)v(\rho)} \\ &= \frac{h(\rho)w(u) + h(u) - h(u)w(\rho)}{1 - v(u) - w(\rho) + v(u)w(\rho) - v(\rho)w(u)} \\ &= \frac{h(\rho)g(u)\psi^T + h(u) - h(u)g(\rho)\psi^T}{1 - f(u) - r(u)\psi^T - g(\rho)\psi^T + f(u)g(\rho)\psi^T - f(\rho)g(u)\psi^T + O(\psi^{2T})} \\ &= \frac{h(u)}{1 - f(u)} + \frac{h(u)}{1 - f(u)} \left( \frac{h(\rho)g(u)}{h(u)} + \frac{r(u) + f(\rho)g(u)}{1 - f(u)} \right) \psi^T + O(\psi^{2T}) \end{aligned}$$

as  $T \rightarrow \infty$ , uniformly for  $u \in D(1, \nu)$ . To keep the notation simple, we have used the abbreviations  $f(y) = f(y, u)$ ,  $g(y) = g(y, u, T)$ ,  $r(y) = r(y, u, T)$  and  $\psi = u/\rho$  as usual. Similarly, we find

$$(1 - w(\rho))H_\rho = \frac{v(\rho)w(u)}{1 - v(u)}H_\rho + \frac{v(\rho)h(u)}{1 - v(u)} + h(\rho)$$

and similarly

$$\begin{aligned} H_\rho &= \frac{h(u)v(\rho) + h(\rho) - h(\rho)v(u)}{1 - v(u) - w(\rho) + v(u)w(\rho) - v(\rho)w(u)} \\ &= h(\rho) + \frac{h(u)f(\rho)}{1 - f(u)} + O(\psi^T) \quad \text{as } T \rightarrow \infty, \end{aligned}$$

uniformly for  $u \in D(1, \nu)$ , too. Note, that we need a weaker asymptotic expansion for  $H_\rho$  only, cf. equation (4.8). Substituting

$$h_k(y_k, u, T) = t_{k+1}(y_{k+1}, u, T)u^{kT}$$

for  $k \geq 0$  and

$$\begin{aligned} f(y_k, u) &= \alpha(y_k, u), \\ g(y_k, u, T) &= -\frac{\gamma(y_k, u)}{2\sqrt{\pi}}T^{-3/2} + O(T^{-2}), \\ r(y_k, u, T) &= O(T^{-2}), \\ h(y_1) &= L(a_0 y_1) \end{aligned}$$

into the recurrence relation of Lemma 4.3 yields (4.7) times  $u^{(k-1)T}$ , of course, and establishes the connection between (4.8) and  $H_u, H_\rho$ :

$$\begin{aligned} \sum_{k \geq 1} t_k(u, u, T)u^{kT} &= u^T H_u, \\ \sum_{k \geq 1} t_k(\rho, u, T)u^{kT} &= u^T H_\rho. \end{aligned} \tag{4.9}$$

It is worth mentioning that, strictly speaking, both remainder terms  $O(T^{-2})$  in the substitution above denote different functions for different indices  $k$ , cf. our remark following equation (4.7). Thus, we rather should have defined functions  $g_k(y_k, u, T)$  and  $r_k(y_k, u, T)$  in our previous treatment. However, all (algebraic) operations required are justified for our simplifying assumption due to the uniform estimations, too.

We obtain

$$H_u = \frac{L(a_0 u)}{1 - \alpha(u, u)} - \frac{L(a_0 u)\gamma(u, u)}{2\sqrt{\pi}(1 - \alpha(u, u))} \left( \frac{L(a_0 \rho)}{L(a_0 u)} + \frac{\alpha(\rho, u)}{1 - \alpha(u, u)} \right) u^T T^{-3/2} \rho^{-T} + O(u^T T^{-2} \rho^{-T})$$

and

$$H_\rho = L(a_0 \rho) + \frac{L(a_0 u)\alpha(\rho, u)}{1 - \alpha(u, u)} + O(u^T T^{-3/2} \rho^{-T}).$$

Mentioning (4.4) and (4.5) it is easy to obtain

$$\frac{1}{1 - \alpha(u, u)} = \left( 1 - \frac{L(B(u)) - L(a_0 u)}{L(B(u))} \right)^{-1} = \frac{L(B(u))}{L(a_0 u)}$$

and

$$\frac{L(a_0\rho)}{L(a_0u)} + \frac{\alpha(\rho, u)}{1 - \alpha(u, u)} = \frac{L(a_0\rho)}{L(a_0u)} + \frac{L(\frac{\rho}{u}B(u)) - L(a_0\rho)}{L(B(u))} \cdot \frac{L(B(u))}{L(a_0u)} = \frac{L(\frac{\rho}{u}B(u))}{L(a_0u)}.$$

In addition, we have

$$\begin{aligned} \gamma(u, u) &= \frac{b(\tau - a_0\rho)}{(1 - \rho/u)(\tau - a_0u)L(\tau)} \left( \frac{L(\tau) - L(a_0u)}{\tau - a_0u} - L'(\tau) - \frac{L(\tau) - L(a_0u)}{\tau - a_0\rho} + \right. \\ &\quad \left. \frac{L'(\tau)}{L(\tau)}(L(\tau) - L(a_0u)) \right) \\ &= \frac{b}{(1 - \rho/u)(\tau - a_0u)L(\tau)} \left( \frac{(L(\tau) - L(a_0u))a_0(u - \rho)}{\tau - a_0u} - \frac{(\tau - a_0\rho)L'(\tau)L(a_0u)}{L(\tau)} \right) \end{aligned}$$

and ultimately

$$H_u = L(B(u)) - \frac{L(B(u))\gamma(u, u)L(\frac{\rho}{u}B(u))}{2\sqrt{\pi}L(a_0u)} u^T T^{-3/2} \rho^{-T} + O(u^T T^{-2} \rho^{-T})$$

and

$$H_\rho = L(\frac{\rho}{u}B(u)) + O(u^T T^{-3/2} \rho^{-T}).$$

Substituting the above in equation (4.8) while mentioning (4.9) yields the desired asymptotic expression for  $B_T(u)$ :

$$\begin{aligned} B_T(u) &= a_0u + B(u) - a_0u - (B(u) - a_0u) \frac{\gamma(u, u)L(\frac{\rho}{u}B(u))}{2\sqrt{\pi}L(a_0u)} u^T T^{-3/2} \rho^{-T} - \\ &\quad \frac{b((\tau - a_0\rho)L'(\tau) - L(\tau))L(\frac{\rho}{u}B(u))}{2\sqrt{\pi}(1 - \rho/u)L(\tau)^2} u^T T^{-3/2} \rho^{-T} + O(u^T T^{-2} \rho^{-T}) \\ &= B(u) - \\ &\quad \frac{bL(\frac{\rho}{u}B(u))}{2\sqrt{\pi}(1 - \rho/u)(\tau - a_0u)L(\tau)} \left( \frac{(B(u) - a_0u)(L(\tau) - L(a_0u))a_0u(1 - \rho/u)}{L(a_0u)(\tau - a_0u)} + \right. \\ &\quad \left. \frac{(\tau - B(u))(\tau - a_0\rho)L'(\tau)}{L(\tau)} - \tau + a_0u \right) u^T T^{-3/2} \rho^{-T} + \\ &\quad O(u^T T^{-2} \rho^{-T}) \end{aligned}$$

**THEOREM 4.5 (ASYMPTOTIC EXPANSION OF  $B_T^{(m)}(1)$ ).** *With the notations above, the first few factorial moments of  $B_T(u)$  have the following asymptotic expansions for  $T \rightarrow \infty$ :*

$$\begin{aligned} B_T(1) &= 1 - \frac{bL(\rho)}{2\sqrt{\pi}(\rho - 1)(\tau - a_0)L(\tau)} \left( \frac{(1 - a_0)(L(\tau) - L(a_0))a_0(\rho - 1)}{L(a_0)(\tau - a_0)} + \right. \\ &\quad \left. \tau - a_0 - \frac{(\tau - 1)(\tau - a_0\rho)L'(\tau)}{L(\tau)} \right) T^{-3/2} \rho^{-T} + O(T^{-2} \rho^{-T}), \end{aligned}$$

$$B_T'(1) = \frac{1}{1 - P'(1)} + O(T^{-1/2} \rho^{-T}),$$

$$B_T^{(m)}(1) = O(1) \quad \text{for } m \text{ arbitrary but fixed}$$

where  $B(z)$  denotes the solution of  $B(z) = zP(B(z))$  and

$$P(z) = A(L(z))$$

$$b = \sqrt{\frac{2P(\tau)}{P''(\tau)}}.$$

**Proof:** The expression for  $B_T(1)$  is straightforward. Since, roughly speaking, the derivation of an asymptotic expression is permitted if its domain of validity lies within the complex plane, the necessary derivatives of  $B_T(u)$  are most easily obtained: As frequently mentioned, we have a remainder term which represents a function analytic for  $u \in D(1, \nu)$ .

The result for  $B'_T(1) = B'(1) + O(T^{-1/2}\rho^{-T})$  follows by differentiating  $B(z) = zP(B(z))$  w. r. t.  $z$ . ■

Note, that the remainder in the asymptotic expressions of  $B_T(1)$  and  $B'_T(1)$  might be improved to  $O(T^{-5/2}\rho^{-T})$  and  $O(T^{-3/2}\rho^{-T})$ , respectively, due to our remark on equation (4.1)!

## 5. FINAL RESULTS

Now we are able to return to the PGF of  $SRD(T)$ , which has been evaluated to

$$S_T(u) = \sum_{k \geq 0} s_{k,T} u^k = \frac{1 - B_T(1)}{1 - B_T(u)},$$

cf. equation (3.2). We shall investigate the moments of this distribution, i.e., the quantities

$$E^n(T) = E[SRD(T)^n] = \sum_{k \geq 0} k^n s_{k,T}.$$

In addition, we define the  $n$ -th factorial moment by

$$F^n(T) = \sum_{k \geq 0} [k]_n s_{k,T} = S_T^{(n)}(1),$$

where  $[k]_n = k(k-1)\cdots(k-n+1)$  denotes the falling factorial. Note, that  $n$  is assumed to be fixed: all  $O()$ -terms are uniform in  $T$  only. Since  $[k]_n = k^n + O(k^{n-1})$ , we obtain

$$F^n(T) = E^n(T) + O(E^{n-1}(T)).$$

If we could provide  $F^{n-1}(T) = O(F^n(T))$ , a simple induction argument shows

$$E^n(T) = F^n(T) + O(F^{n-1}(T)), \tag{5.1}$$

hence it seems reasonable to investigate the factorial moments. We have

$$S_T(z) = g(B_T(z))$$



for  $g(z) = (1 - B_T(1))/(1 - z)$ . An easy computation shows

$$g^{(j)}(z)|_{z=B_T(1)} = g^{(j)}(B_T(1)) = \frac{j!}{(1 - B_T(1))^j}$$

for all  $j \geq 0$ . Using the formula of Faà di Bruno (cf. [5, p. 50])

$$(b(a(z)))^{(n)} \Big|_{z=t} = \sum_{j=0}^n b^{(j)}(a(t)) \sum_{\substack{k_1+k_2+\dots+k_n=j \\ k_1+2k_2+\dots+nk_n=n \\ k_i \geq 0}} \frac{n!}{k_1!(1!)^{k_1} \dots k_n!(n!)^{k_n}} (a^{(1)}(t))^{k_1} \dots (a^{(n)}(t))^{k_n},$$

we are able to express  $S_T^{(n)}(1)$  in terms of  $g^{(j)}(B_T(1))$  and  $B_T^{(j)}(1)$ ; setting  $b(z) = g(z)$ ,  $a(z) = B_T(z)$ , and  $t = 1$ , we find

$$S_T^{(n)}(1) = \sum_{j=0}^n \frac{1}{(1 - B_T(1))^j} \sum_{\substack{k_1+\dots+k_n=j \\ k_1+\dots+nk_n=n \\ k_i \geq 0}} c_{j,n,k_1,k_2,\dots,k_n} (B_T^{(1)}(1))^{k_1} \dots (B_T^{(n)}(1))^{k_n}$$

with the abbreviation

$$c_{j,n,k_1,k_2,\dots,k_n} = j! \frac{n!}{k_1!(1!)^{k_1} \dots k_n!(n!)^{k_n}}.$$

Using the fact  $B_T^{(m)}(1) = O(1)$  for  $m \geq 0$  from Theorem 4.5, an overall contribution of the inner sum of  $O(1)$  may be found. Because of

$$1 - B_T(1) = O(T^{-3/2} \rho^{-T})$$

the major contributions come from  $(1 - B_T(1))^{-j}$  with  $j = n$ . Hence we may discard all terms of the outer sum concerning  $S_T^{(n)}(1)$  except for  $j = n$ , i.e., we obtain

$$\begin{aligned} S_T^{(n)}(1) &= \sum_{\substack{k_1+\dots+k_n=n \\ k_1+\dots+nk_n=n \\ k_i \geq 0}} \frac{c_{n,n,k_1,k_2,\dots,k_n}}{(1 - B_T(1))^n} (B_T^{(1)}(1))^{k_1} \dots (B_T^{(n)}(1))^{k_n} + O(T^{3(n-1)/2} \rho^{(n-1)T}) \\ &= n! \left( \frac{B_T^{(1)}(1)}{1 - B_T(1)} \right)^n + O(T^{3(n-1)/2} \rho^{(n-1)T}), \end{aligned} \quad (5.2)$$

since the conditions concerning the inner sum hold for  $k_1 = n$  only. Substituting the expansion above, we find

$$F^n(T) = S_T^{(n)}(1) = O(T^{3n/2} \rho^{nT}).$$

Since  $F^{n-1}(T) = O(F^n(T))$ , the condition for equation (5.1) is justified and we may conclude

$$E^n(T) = F^n(T) + O(F^{n-1}(T)).$$

The remainder above disappears within the remainder term established for  $S_T^{(n)}(1)$ , hence our final result follows:

THEOREM 5.1. With the conditions (i)–(vi) from Section 4 and the notations above, the  $n$ -th moment ( $n$  arbitrary but fixed) of  $\text{SRD}(T)$  fulfills

$$E^n(T) = n! \mu(T)^n (1 + O(T^{-1/2}))$$

with

$$\mu(T) = \frac{2\sqrt{\pi}(\rho-1)(\tau-a_0)L(\tau)}{bL(\rho)(1-P'(1))} \left( \frac{(1-a_0)(L(\tau)-L(a_0))a_0(\rho-1)}{L(a_0)(\tau-a_0)} + \right. \\ \left. \tau - a_0 - \frac{(\tau-1)(\tau-a_0\rho)L'(\tau)}{L(\tau)} \right)^{-1} T^{3/2} \rho^T$$

where  $P(z) = A(L(z))$ ,  $\rho = \frac{\tau}{P(\tau)}$  and  $b = \sqrt{\frac{2P(\tau)}{P''(\tau)}}$ .

Note, that the remainder  $1 + O(T^{-1/2})$  springs from  $(\frac{B'_T(1)}{1-B_T(1)})^n$ ; it causes the remainder of (5.2) to disappear.

## CONCLUSIONS

This paper contains a detailed analysis of the successful run duration  $\text{SRD}(T)$  of a discrete time single server system with nonpreemptive LCFS task scheduling.  $\text{SRD}(T)$  is closely related to the ability of this system to meet the fixed deadlines  $T$  of all tasks arriving at the system, from the time it is turned on to the year 9999, for example. It extends our analysis of preemptive LCFS scheduling (cf. [2]) and FCFS scheduling (cf. [8]) to the case of the nonpreemptive LCFS scheduling discipline. Again, we have found impressive results concerning the expectation of  $\text{SRD}(T)$ , unfortunately weakened by a large standard deviation; see [8] for a more detailed discussion.

Comparing nonpreemptive LCFS to FCFS scheduling shows a significantly better deadline meeting behaviour of the latter. On the other hand, nonpreemptive LCFS and preemptive LCFS is more difficult to compare; it is devoted to a forthcoming paper. Note however, that those results are the same for  $L(z) = z$ , i.e., constant task execution times of 1 cycle.

To establish our results we have used a coefficient extraction technique for multivariate functions which we call *asymptotic separation*: using a slight extension of well-known asymptotic techniques it is possible to separate multivariate analytic functions. We feel that this method is of independent interest and should be useful in the case of investigating sequences of random variables  $X_k$ , which are in some sense “weakly dependent”. In our case, we had to deal with random variables having a Markov-like property, but asymptotic separation is not restricted to this case. Note however, that queueing theory provides no solution to our problem, because we are forced to study non-equilibrium behaviour in order to obtain our desired quantities.

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