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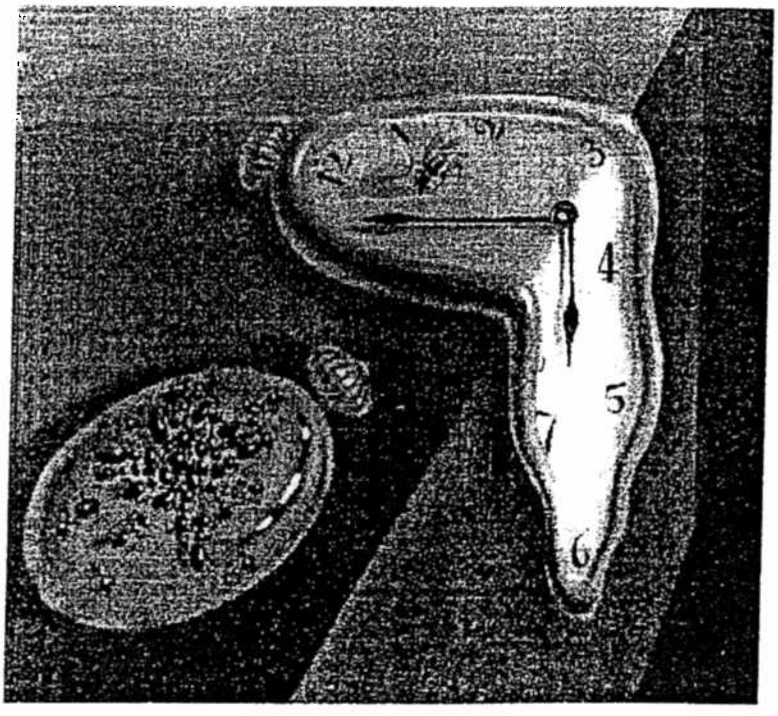

Ausschnitt aus: Salvador Dali, "Die Beständigkeit der Erinnerung"

The Analysis of the Expected Successful Operation Time of Slotted ALOHA

M. DRMOTA[1] AND U. SCHMID[2]

Abstract. It has been well-known for nearly twenty years that the bistable behavior of infinite population slotted ALOHA networks causes the unpleasant effect of eventually reaching a overloaded state, where the number of backlogged stations becomes larger and larger and the useful throughput reduces to zero. The detailed analysis reveals that this statement is true for any average offered load $\lambda > 0$, regardless of the retransmission probability p. A challenging and to the best of our knowledge not sufficiently solved problem within this context concerns the time taken until this destabilization occurs. We succeeded in answering this question based on the fact that the operation of the system may be viewed as a sequence of consecutive busy periods, each starting from backlog 0 and returning to backlog 0. It turns out that the whole period of successful operation $\mathcal{S}$ consists of a finite sequence of busy periods of finite lengths, which is "terminated" by an infinite busy period (which never returns to backlog 0). A further analysis of this simple renewal process leads to an infinite dimensional system of linear equations, which is shown to have only one meaningful solution. A pair of upper and lower asymptotic bounds for that solution eventually provide the key to our major result, namely the average number of slots up to the beginning of the infinite busy period $\mathrm{E}[\mathcal{S}] = \exp\left(\frac{\log^2 \lambda^{-1}}{-2\log(1-p)} + \frac{\log \lambda^{-1} \log\log \lambda^{-1}}{-\log(1-p)} + O\left(\frac{\log \lambda^{-1}}{-\log(1-p)}\right)\right)$ uniformly for $p \to 0$ and $\lambda \to 0$.

Keywords: collision resolution algorithms, slotted ALOHA, transient Markov chains, busy periods, asymptotic analysis.

1. INTRODUCTION

It has been more than 20 years since N. Abramson developed his famous paradigm for multiuser communication networks, the ALOHA system of the University of Hawaii. Since then, a considerable amount of research has been devoted to the analysis, the improvement, and the generalization of such collision resolution algorithms for contention-based broadcast networks; an overview may be found in [**GA**]. However, despite the development of certain non-ALOHA algorithms offering very much better characteristics, the ALOHA algorithm is still important due to its simplicity; even Ethernet uses a modification of the scheme.

A well-known variety of original ALOHA is the slotted ALOHA algorithm, which works as follows: Consider an infinite population of identical stations sharing a single time-slotted communication channel. Data is transmitted in form of fixed-size packets fitting into exactly one slot. The whole population generates new data packets according to a Poisson process with an overall rate λ. Each station generating a new packet transmits it immediately during the very next slot. If more than one station transmits during one and the same slot, a collision occurs and all packets involved become garbled and therefore lost; this occurrence may be detected using certain checksumming methods. Now, if a station

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has been involved in a collision, it transmits its data packet during each subsequent slot with a fixed probability p until a successful transmission of the packet occurs. Note that a station being in this *backlogged* state is assumed to be unable to generate new packets.

Denoting the backlog at slot k, that is, the number of stations being in the backlogged state at the beginning of slot k, by $\mathcal{N}_k$, it is clear that $\{\mathcal{N}_k;\ k = 0, 1, \dots\}$ is a homogeneous Markov chain with (stationary) transition probabilities $p_{n,m} = \mathrm{P}[\mathcal{N}_{k+1} = m | \mathcal{N}_k = n]$ and $\mathcal{N}_0 = 0$. The transition probabilities are independent of the "time" k and follow most easily from the model above:

$$\begin{aligned} p_{n,n-j} &= 0 \qquad \text{for } j \geq 2, \\ p_{n,n-1} &= np(1-p)^{n-1}e^{-\lambda}, \\ p_{n,n} &= (1-p)^n \lambda e^{-\lambda} + \big(1 - np(1-p)^{n-1}\big)e^{-\lambda}, \\ p_{n,n+1} &= \big(1 - (1-p)^n\big)\lambda e^{-\lambda}, \\ p_{n,n+j} &= \frac{\lambda^j}{j!}e^{-\lambda} \qquad \text{for } j \geq 2. \end{aligned} \tag{1.1}$$

For example, the equation for $p_{n,n}$, where the backlog remains unaltered by newly generated packets and/or successful transmissions during a single slot, follows from these two possibilities (1) none of the n backlogged stations transmits and exactly one new packet has been generated during the previous slot, and (2) zero or more than one of the n backlogged stations transmit and no new packet has been generated during the previous slot.

The usual analysis of the behavior of slotted ALOHA relies on the investigation of this Markov chain, cf. for example [**KL**]. In [**FGLB**], it was shown that this particular chain is nonergodic and therefore unstable in the following sense: $\mathrm{P}[\mathcal{N}_k < n] \to 0$ for $k \to \infty$ for any finite n. Alternative proofs concerning the nonergodicity of $\{\mathcal{N}_k\}$ may be found in [**FGL**] and [**K**], too. The clearer result that the Markov chain is *transient*[3] and therefore $\mathrm{P}[\lim_{k\to\infty} \mathcal{N}_k = \infty] = 1$ was first derived in [**RT**] by means of a martingale approach. It also follows as a byproduct from an interesting result in [**KE**], namely that the total number of successful transmissions during the operation of slotted ALOHA is finite with probability 1.

Intuitively, this unstable behaviour becomes clear by considering the drift

$$D_n = \mathrm{E}[\mathcal{N}_{k+1} - \mathcal{N}_k | \mathcal{N}_k = n] = \lambda - \big((1-p)^n \lambda e^{-\lambda} + np(1-p)^{n-1}e^{-\lambda}\big), \tag{1.2}$$

which is positive for all $n \geq n_1$; for large $\lambda > e^{-1}$ it is not hard to establish that $D_n > 0$ for all $n > 1$ (if p is sufficiently small). Anyway, for reasonable values of $\lambda < e^{-1}$ and p there is a region of backlogs $n_0 \leq n \leq n_1$ where $D_n < 0$, and the size of this range increases as λ decreases and as p decreases. Consequently, n_0 might be considered as an "attractive" stable operation point, and n_1 as a "critical" backlog where destabilization is almost inevitable.

Naturally, one is interested in quantifying the implications of instability in the operation of slotted ALOHA. One obvious approach in attacking this problem, as pursued in [**KL**],

[3]Note that nonergodicity, as established in [**K**], does not necessarily imply transience due to the possibility of null recurrence.

is to partition the possible states into two regions, the safe region $\{0, 1, \ldots, N\}$ and the unsafe region $\{N+1, N+2, \ldots\}$. A reasonable idea for choosing the (constant) value N is the critical backlog n_1, where the drift D_{n_1} becomes non-negative again. An interesting parameter within this context is the average first entrance time (FET) into the unsafe region, which has been derived in [**KL**] by analyzing a modified Markov chain with an absorbing state for all backlogs larger than N.

However, there are several difficulties with (the analysis of) models based on such ideas (cf. [**CH**], [**GW**] for different examples). The first point of criticism concerns the choice of N, because there is always a small but non-zero probability that it is able to return into the safe region of small backlogs. The major difficulty, however, concerns the appropriate analysis. With traditional approaches ([**KL**], [**CH**]), closed form solutions are usually not available. Moreover, the numerical evaluation of results like FET requires the solution of a large (i.e., infinite) dimensional system of linear equations, which becomes computationally intractable when λ and p are small.

An alternative analysis is based on large deviation methods (cf. [**GW**]), which asymptotically describe the behavior of the Markov chain for small p. The appropriate results show that the system will operate near the stable operation point n_0 for a long period $\mathcal{L}$ of time, after that the process begins to follow an almost *deterministic* path out of the safe region $n \leq n_1$. $\mathcal{L}$ is approximately exponentially distributed with a mean of

$$\mathrm{E}[\mathcal{L}] = e^{c(\lambda)/p + o(1/p)} \qquad \text{as } p \to 0, \tag{1.3}$$

where $c(\lambda)$ is expressible via a certain integral which is easily solved by some numerical[4] methods. The major deficit of large deviation techniques, however, lies in the fact that some of the results have not been completely proven yet.

This paper presents a fundamentally different approach for investigating the period of successful operation of slotted ALOHA, which was stimulated by some of our research on dynamic task scheduling in hard real-time systems (cf. [**SB**], for example). Our — completely proven— technique explicitely avoids the use of large deviation methods and provides an analytic result for small p and λ. The derivations are organized as follows: Section 2 contains some preliminaries and the statement of our major theorems, Section 3 is devoted to considerations concerning the existence of a solution of an infinite dimensional system of linear equations. Sections 4, 5 and 6 contain the derivation of (matching) upper and lower asymptotic bounds for the expected duration of the successful operation of slotted ALOHA, which establish our major results. Finally, some conclusions are summarized in Section 7.

2. Preparations and Major Results

Consider our slotted ALOHA system starting from an idle state, e.g. after turning the power on. The very first slot is obviously bound to be an idle slot. Depending on the

[4] Actually, we are grateful to an anonymous referee for pointing out that our major result $\mathrm{E}[\mathcal{S}]$ (asymptotically) coincides with the large deviation result $\mathrm{E}[\mathcal{L}]$ obtained by an asymptotic evaluation of the integral above; we give a sketch of the idea in our concluding Section 7.

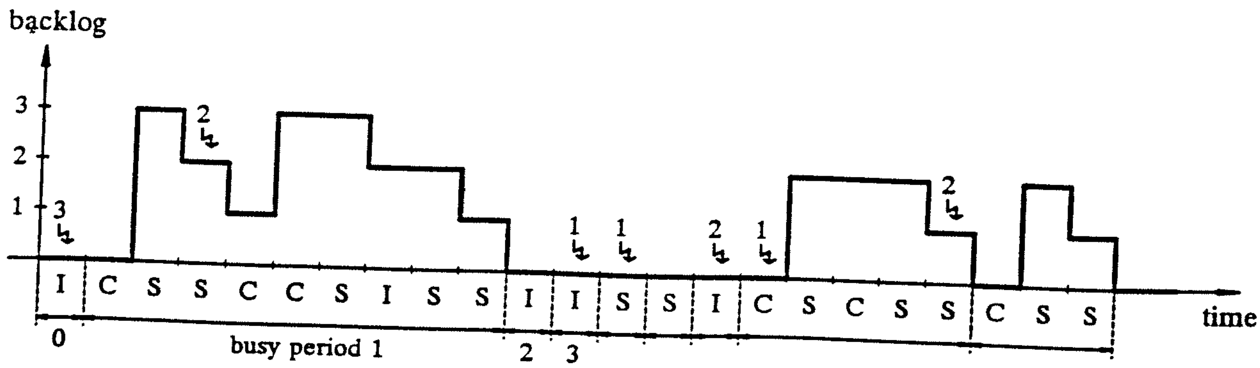
number n of new packets generated during that slot, we have to distinguish between 3 cases:

(1) $n = 0$ new packets
In this case, an idle slot follows, and the backlog at the beginning of the next slot is 0 again.

(2) $n = 1$ new packets
Here, a slot with a successful transmission follows, and the backlog at the beginning of the next slot is 0 again.

(3) $n \geq 2$ new packets
In this case, a collision slot follows, which is caused by the simultaneous transmission attempts of the n stations, bear in mind the description of slotted ALOHA in Section 1. Hence, at the beginning of the next slot, we observe a backlog n, and the slotted ALOHA collision resolution algorithm carries on according to (1.1). However, despite the inherent instability of ALOHA mentioned in Section 1, there is a non-zero probability that our system reaches backlog 0 again.

In all cases, we (probably) reach a renewal point, which enables us to repeat our previous expositions. The role of the very first idle slot mentioned above is taken by the last slot of the previous period. That is, the operation of our system may be viewed as a sequence of consecutive, mutually independent and identically distributed *busy periods*, each starting with backlog 0 and returning to backlog 0. The following diagram provides an example:



The generation of n new packets during a slot is represented by a small lightning symbol with n above. Each slot is marked with I ... idle, S ... successful transmission, C ... collision, respectively.

Letting aside the trivial varieties of busy periods (1) and (2) for a moment, we focus our interest on the non-trivial case (3), where an initial collision slot causes an initial backlog $n \geq 2$. This backlog is handled by a subsequent resolution process $\{\mathcal{N}_k^n;\ k = 0, 1, \dots\}$, $n \geq 1$, which is very similar to the process $\{\mathcal{N}_k\}$ from Section 1. It relies on the same transition probabilities (1.1), the only differences concern the state corresponding to backlog 0, this time representing an absorbing state, and the initial state $\mathcal{N}_0^n = n$.

Let the random variable $\mathcal{Q}_n$, $n \geq 1$ denote the number of slots necessary for $\{\mathcal{N}_k^n\}$ to reach the absorbing state (backlog 0), i.e., the length of the collision resolution interval necessary for (completely) resolving an initial collision of n stations. Note that this *CRI-length* is crucial in the analysis of certain non-ALOHA collision resolution algorithms, cf. [**MF**] for details. Defining the probabilities

$$q_{n,i} = \mathrm{P}[\mathcal{Q}_n = i] \tag{2.1}$$

and the corresponding probability generating function (PGF)

$$Q_n(z) = \sum_{i \geq 0} q_{n,i} z^i, \tag{2.2}$$

we obtain

$$\begin{aligned} q_{n,0} &= 0 \qquad \text{for } n \geq 1, \\ q_{1,1} &= pe^{-\lambda}, \\ q_{1,i+1} &= \sum_{j \geq 1} p_{1,j} q_{j,i} \qquad \text{for } i \geq 1, \\ q_{n,i+1} &= \sum_{j \geq n-1} p_{n,j} q_{j,i} \qquad \text{for } n \geq 2,\ i \geq 0. \end{aligned} \tag{2.3}$$

Multiplying (2.3) by z^{i+1} and summing up, we obtain by recalling definition (1.1)

$$\begin{aligned} Q_1(z) &= ze^{-\lambda}\Big(p + (1-p)(1+\lambda)Q_1(z) + \lambda p Q_2(z) + \sum_{j \geq 2} \frac{\lambda^j}{j!} Q_{j+1}(z)\Big) \\ Q_n(z) &= ze^{-\lambda}\Big(np(1-p)^{n-1}Q_{n-1}(z) + \big(\lambda(1-p)^n + 1 - np(1-p)^{n-1}\big)Q_n(z) \\ &\qquad + \lambda\big(1-(1-p)^n\big)Q_{n+1}(z) + \sum_{j \geq 2} \frac{\lambda^j}{j!} Q_{n+j}(z)\Big) \quad \text{for } n \geq 2. \end{aligned} \tag{2.4}$$

Now we are ready to write down the PGF of the length of a busy period of arbitrary type. Defining

$$\beta_k = \mathrm{P}[\text{Length of a busy period is exactly } k \text{ slots}], \tag{2.5}$$

we easily obtain

$$B(z) = \sum_{k \geq 0} \beta_k z^k = e^{-\lambda} z + \lambda e^{-\lambda} z + e^{-\lambda} z \sum_{j \geq 2} \frac{\lambda^j}{j!} Q_j(z). \tag{2.6}$$

Note the first and the second term, which represent the trivial varieties (1) and (2) of busy periods.

Remembering the fact that the Markov chain for slotted ALOHA $\{\mathcal{N}_k\}$ is transient (cf. our remarks in Section 1), we have by definition

$$B(1) = \mathrm{P}[\exists k > 0 \text{ such that } \mathcal{N}_k = 0] < 1. \tag{2.7}$$

Thus, the period of successful operation $\mathcal{S} = \max\{k | \mathcal{N}_k = 0\}$ of a slotted ALOHA system consists of a *finite* number of busy periods of finite length, terminated by an *infinite* busy period of destabilization. Defining $s_k = \mathrm{P}[\mathcal{S} = k]$, the appropriate PGF evaluates to

$$S(z) = \sum_{k \geq 0} s_k z^k = \frac{1 - B(1)}{1 - B(z)}. \tag{2.8}$$

This follows from the fact that the PGF of the length of an arbitrary number of finite busy periods is $\sum_{n \geq 0} B(z)^n$, and that the probability of the occurrence of the terminating infinite busy period equals $1 - B(1)$.

It is easy to derive the desired expectation of the successful operation of slotted ALOHA from equation (2.8). Differentiating and substituting $z = 1$ yields

$$\mathrm{E}[\mathcal{S}] = \frac{B'(1)}{1 - B(1)}. \tag{2.9}$$

Our problem is therefore reduced to the computation of

$$\begin{aligned} 1 - B(1) &= 1 - e^{-\lambda}\Big(1 + \lambda + \sum_{j \geq 2} \frac{\lambda^j}{j!} Q_j(1)\Big) \\ &= e^{-\lambda} \sum_{j \geq 2} \frac{\lambda^j}{j!} (1 - Q_j(1)) \end{aligned} \tag{2.10}$$

and

$$B'(1) = B(1) + e^{-\lambda} \sum_{j \geq 2} \frac{\lambda^j}{j!} Q'_j(1), \tag{2.11}$$

which leads to the problem of solving the infinite dimensional system of equations (2.4) for $Q_n(z)$ and $Q'_n(z)$, $n \geq 1$ at $z = 1$. Our first aim is to show that we are concerned with several different solutions, cf. Section 3. These considerations lead to an elegant direct proof of the already known

THEOREM 2.1 (TRANSIENT MARKOV CHAIN $\mathcal{N}_k$). *The Markov chain for slotted ALOHA $\{\mathcal{N}_k;\ k = 0, 1, \dots\}$, $\mathcal{N}_0 = 0$ and transition probabilities (1.1) is transient.*

After determining the (unique) solution of interest, we provide an upper and a lower bound for $Q_n(1)$ and therefore $1/(1 - B(1))$ in Sections 4 and 5, respectively; Section 6 is devoted to the bounds for $Q'_n(1)$ and hence $B'(1)$. The appropriate asymptotic analysis reveals that the resulting bounds for $\mathrm{E}[\mathcal{S}]$ are equivalent up to some lower order term, finally establishing

THEOREM 2.2 (ASYMPTOTICS OF $\mathrm{E}[\mathcal{S}]$). *The expected duration of the successful operation of slotted ALOHA, which is the average number of slots up to the beginning of the busy period of destabilization, is*

$$\mathrm{E}[\mathcal{S}] = \exp\Big(\frac{\log^2 \lambda^{-1}}{-2\log(1-p)} + \frac{\log \lambda^{-1} \log\log \lambda^{-1}}{-\log(1-p)} + O\big(\frac{\log \lambda^{-1}}{-\log(1-p)}\big)\Big)$$

uniformly[5] *for* $p \to 0$ *and* $\lambda \to 0$; $\exp(x) = e^x$ *denotes the exponential function.*

For notational convenience, we finally introduce the abbreviations

$$\begin{aligned} Q_n &= Q_n(1), \\ Q'_n &= Q'_n(1), \\ B &= B(1), \\ B' &= B'(1) \end{aligned} \tag{2.12}$$

to be used in the subsequent sections.

3. On the Existence of a Solution

With the beginning of this section, we will change our point of view from an application-oriented to a pure mathematical one. We focus our attention on the solution(s) of the infinite dimensional system (2.4) of linear equations for $Q_n = Q_n(1)$, $n \geq 1$, which can be reformulated in terms of a fixed point problem of an affine mapping.

Consider the Banach space $\mathbf{l}^\infty$, the space of all bounded sequences[6] $\mathbf{x} = (x^{(i)})_{i\geq 1}$ with norm $\|\mathbf{x}\| = \|\mathbf{x}\|_\infty = \sup_{i\geq 1} |x^{(i)}|$ and define

$$F : \mathbf{l}^\infty \mapsto \mathbf{l}^\infty, \quad \mathbf{x} \mapsto \mathbf{L}\mathbf{x} + \mathbf{c}, \tag{3.1}$$

where

$$(\mathbf{L}\mathbf{x})^{(i)} = \sum_{m\geq 1} p_{i,m} x^{(m)}, \qquad i \geq 1 \tag{3.2}$$

and

$$c^{(i)} = p_{i,0} = \begin{cases} pe^{-\lambda}, & \text{for } i = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

(Note that the $p_{i,m}$ are the same of Section 1.) Then $\mathbf{Q} = (Q_n)_{n\geq 1} = (Q_n(1))_{n\geq 1}$ is apparently a fixed point of $F(\mathbf{x}) = \mathbf{x}$.

To ask for solutions is of vital importance, since $x^{(i)} = 1$ for all $i \geq 1$ is obviously a solution of $F(\mathbf{x}) = \mathbf{x}$. However, remembering the inherent instability of slotted ALOHA, one would rather expext a solution with $\lim_{n\to\infty} Q_n = 0$. Indeed we will show that there is (at most) one solution for every $\lim_{n\to\infty} Q_n = Q$ fixed a priori.

The general plan of our subsequent derivations is as follows: Introducing the notations $\mathbf{x} \leq \mathbf{y}$ if $x^{(i)} \leq y^{(i)}$ for all $i \geq 1$, $\mathbf{l}^\infty(l)$ for the set of sequences $\mathbf{x} = (x^{(i)})_{i\geq 1}$ with $\lim_{i\to\infty} x^{(i)} = l$, and $\mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]$ for those sequences $\mathbf{x} \in \mathbf{l}^\infty$ with $\mathbf{a} \leq \mathbf{x} \leq \mathbf{b}$, our essential aim is to find sequences $0 \leq \mathbf{a} \leq \mathbf{b} \in \mathbf{l}^\infty(0)$ such that $F(\mathbf{a}) \geq \mathbf{a}$ and $F(\mathbf{b}) \leq \mathbf{b}$ (cf. sections 4 and 5), which ensure that there are solutions $\mathbf{x} \in \mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]$ of $F(\mathbf{x}) = \mathbf{x}$ (Lemma 3.3). Assuming the existence of such sequences $\mathbf{a}, \mathbf{b} \in \mathbf{l}^\infty(0)$, it can be shown that our solution $(Q_n)_{n\geq 1}$ of $F(\mathbf{x}) = \mathbf{x}$ satisfies $\lim_{n\to\infty} Q_n = 0$ (Lemma 3.6). However, since there is at

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[5] The phrase *uniformly* for $p \to 0$ and $\lambda \to 0$ means that there exist p_0, λ_0 such that the asymptotic expansion (i.e., the remainder term) holds for any $0 < p \leq p_0$ and $0 < \lambda \leq \lambda_0$. Therefore, our result can be applied to some fixed $\lambda > 0$ and $p \to 0$ and for some fixed $p > 0$ and $\lambda \to 0$ as well.

[6] We will use the notation $\mathbf{x} = (x^{(i)})_{i\geq 1}$ or $\mathbf{x} = (x_i)_{i\geq 1}$ interchangeably.

most one solution $\mathbf{x} \in \mathbf{l}^\infty(0)$ of $F(\mathbf{x}) = \mathbf{x}$ (Lemma 3.2), $\mathbf{Q} = (Q_n)_{n\geq 1}$ must be contained in $\mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]$. The (known) upper and lower bounds $\mathbf{b}$, $\mathbf{a}$ for Q_n, however, will enable us to get asymptotic upper and lower bounds for $1/(1 - B(1))$.

Now we return to our mathematical considerations. It is worth noting that a mapping of the kind $F(\mathbf{x}) = \mathrm{L}\mathbf{x} + \mathbf{c}$ has a unique fixed point $\mathbf{x} \in \mathbf{l}^\infty$ if $\|\mathrm{L}\| = \sup_{\|\mathbf{x}\|\leq 1} \|\mathrm{L}\mathbf{x}\| < 1$ by Banach's fixed point theorem. Unfortunately, we have $\|\mathrm{L}\| = 1$ here because of $\sum_{m\geq 1} p_{i,m} = 1$ for $i > 1$ (and $\sum_{m\geq 1} p_{1,m} < 1$). Consequently, the inequality $\|F(\mathbf{x}) - F(\mathbf{y})\| = \|\mathrm{L}\mathbf{x} - \mathrm{L}\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$ cannot be generally improved. However, if we restrict ourselves to the subset $\mathbf{l}^\infty(\ell) \subseteq \mathbf{l}^\infty$ of all convergent sequences $\mathbf{x} = (x^{(i)})_{i\geq 1}$ with $\lim_{i\to\infty} x^{(i)} = \ell$, it is not hard to establish

Lemma 3.1 (Contraction Property). *For every $\mathbf{x} \neq \mathbf{y}$ in $\mathbf{l}^\infty(\ell)$, the mapping $F(\mathbf{x}) = \mathrm{L}\mathbf{x} + \mathbf{c}$ with L according to (3.2) and $\mathbf{c}$ arbitrary provides*

$$\|F(\mathbf{x}) - F(\mathbf{y})\| = \|\mathrm{L}\mathbf{x} - \mathrm{L}\mathbf{y}\| < \|\mathbf{x} - \mathbf{y}\|.$$

Proof: It is sufficient to show $\|\mathrm{L}\mathbf{x}\| < \|\mathbf{x}\|$ for every sequence $\mathbf{x}$ with $\|\mathbf{x}\| > 0$ and $\lim_{i\to\infty} |x^{(i)}| = 0$. Now, the last condition ensures that for any $\varepsilon > 0$ there is an $I(\varepsilon)$, such that $|x^{(i)}| < \varepsilon$ for $i > I(\varepsilon)$. Setting $\varepsilon = \|\mathbf{x}\|/2$, we have for $i > I(\varepsilon) + 1$

$$\left|(\mathrm{L}\mathbf{x})^{(i)}\right| = \Big|\sum_{m=i-1}^{\infty} p_{i,m} x^{(m)}\Big| < \varepsilon = \|\mathbf{x}\|/2,$$

since $p_{i,m} = 0$ for $m < i - 1$. On the other hand, for $i \leq I(\varepsilon) + 1$ we have the bound

$$\left|(\mathrm{L}\mathbf{x})^{(i)}\right| < \|\mathbf{x}\|,$$

which follows from the obvious fact that $|x^{(m)}| = \|\mathbf{x}\|$ for finitely many m only, according to $\lim_{i\to\infty} x^{(i)} = 0$, and $p_{i,m} > 0$ for $m \geq i - 1$.

Thus, we obtain

$$\|\mathrm{L}\mathbf{x}\| \leq \max\Big(\left|(\mathrm{L}\mathbf{x})^{(1)}\right|, \ldots, \left|(\mathrm{L}\mathbf{x})^{(I(\varepsilon)+1)}\right|, \|\mathbf{x}\|/2\Big) < \|\mathbf{x}\|,$$

which completes the proof of Lemma 3.1. ∎

Now it is easy to get

Lemma 3.2 (Uniqueness). *Let l be real and fixed. Then there exists at most one solution $\mathbf{x} = (x^{(i)})_{i\geq 1} \in \mathbf{l}^\infty(l)$ of $F(\mathbf{x}) = \mathbf{x}$.*

Proof: If there are two different fixed points $\mathbf{x}, \mathbf{y} \in \mathbf{l}^\infty(Q)$ of the mapping (3.1), then $0 < \|\mathbf{x} - \mathbf{y}\| = \|\mathrm{L}\mathbf{x} + \mathbf{c} - \mathrm{L}\mathbf{y} - \mathbf{c}\| < \|\mathbf{x} - \mathbf{y}\|$ yields a contradiction. ∎

In order to prove the existence of solutions, we reformulate Theorem I of Chapter 2 in [**KK**] (also compare with [**GMB**] and [**MB**]):

LEMMA 3.3 (EXISTENCE). *Let* $\mathbf{a}, \mathbf{b} \in \mathbf{l}^\infty$ *be two sequences satisfying* $\mathbf{0} \le \mathbf{a} \le \mathbf{b}$, $F(\mathbf{a}) \ge \mathbf{a}$, *and* $F(\mathbf{b}) \le \mathbf{b}$. *Then there are solutions* $\mathbf{x} \in \mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]$ *of* $F(\mathbf{x}) = \mathbf{x}$.

Proof: Set $\mathbf{a}_1 = \mathbf{a}$ and $\mathbf{a}_{n+1} = F(\mathbf{a}_n)$ for $n \ge 1$. Then it is clear that $\mathbf{a}_n \le \mathbf{a}_{n+1} \le \mathbf{b}$, i.e., the real sequences $(a_n^{(i)})_{n\ge 1}$ monotonically increase and are bounded above for all $i \ge 1$. Setting $x^{(i)} = \lim_{n\to\infty} a_n^{(i)}$ and $\mathbf{x} = (x^{(i)})_{i\ge 1}$, Lebesgue's theorem on dominated convergence establishes

$$\begin{aligned} x^{(i)} &= \lim_{n\to\infty} a_{n+1}^{(i)} \\ &= \lim_{n\to\infty} \sum_{m\ge 1} p_{i,m} a_n^{(m)} + p_{i,0} \\ &= \sum_{m\ge 1} p_{i,m} x^{(m)} + p_{i,0}. \end{aligned}$$

Therefore, $\mathbf{x} \in \mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]$ is a solution of $F(\mathbf{x}) = \mathbf{x}$. ∎

Combining Lemmata 3.2 and 3.3 we obtain

LEMMA 3.4 (EXISTENCE AND UNIQUENESS$_1$). *Suppose that there are sequences* $\mathbf{a}, \mathbf{b} \in \mathbf{l}^\infty(\ell)$ *with* $\mathbf{0} \le \mathbf{a} \le \mathbf{b}$, *such that the mapping* $F(\mathbf{x}) = \mathrm{L}\mathbf{x} + \mathbf{c}$ *with* L *according to (3.2) and* $\mathbf{c}$ *arbitrary satisfies* $F(\mathbf{a}) \ge \mathbf{a}$ *and* $F(\mathbf{b}) \le \mathbf{b}$. *Then there exists a unique fixed point* $\mathbf{x} \in \mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]$ *of* $F(\mathbf{x})$.

It should be noted that Lemma 3.3 can be interpreted from a more topological point of view considering only subsets $\mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]$ with $\mathbf{0} \le \mathbf{a} \le \mathbf{b}$ and $\mathbf{a}, \mathbf{b} \in \mathbf{l}^\infty(l)$ for some fixed real l. It is not hard to verify that in this case $\mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]$ is a compact subset of the Banach space $\mathbf{l}^\infty$. Furthermore, if $F(\mathbf{a}) \ge \mathbf{a}$ and $F(\mathbf{b}) \le \mathbf{b}$ then $F(\mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]) \subseteq \mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]$, where F is obviously a continuous mapping. Therefore, we can restrict F to $\mathbf{l}^\infty[\mathbf{a}, \mathbf{b}]$ and can apply the following easy

PROPOSITION 3.5 (EXISTENCE AND UNIQUENESS$_2$). *Let* (X, d) *be a compact metric space and* $f : X \mapsto X$ *a continuous function satisfying* $d\big(f(x), f(y)\big) < d(x, y)$ *for* $x \ne y$. *Then,* f *has a unique fixed point.*

Proof: Set $X_1 = X$ and $X_{n+1} = f(X_n)$ for $n \ge 1$. Then $X_n \ne \emptyset$ is closed and $X_{n+1} \subseteq X_n$. According to the finite intersection property of closed subsets of compact spaces, it follows that $Y = \bigcap_{n\ge 1} X_n \ne \emptyset$, and of course all (possible) fixed points of $f(x)$ are contained in Y. Trivially, $f(Y) \subseteq Y$. On the other hand, for every $y \in Y$ the subsets $Y_k = X_k \cap f^{-1}(\{y\})$ are closed and satisfy the finite intersection property. Thus $Y \cap f^{-1}(\{y\}) \ne \emptyset$ and so $Y \subseteq f(Y)$. Now $d(f(x), f(y)) < d(x, y)$ for $x \ne y$ immediately shows that Y contains exactly one point which must be the only fixed point of $f(x)$. ∎

Our next aim is to justify the intuitively clear limiting value $\lim_{n\to\infty} Q_n = 0$ for the desired solution. We make use of the following

THEOREM [**FE1**, P. 403]. *If the state space* $S = T \cup R$ *of a Markov chain with transition probabilities* $p_{n,m}$ *consists of a finite set of recurrent states* R *and a set of transient states* T, *the probabilities y_n of ultimate absorption in R when starting from state $n \in T$ are given by the minimal non-negative solution of*

$$y_n = \sum_{m \in T} p_{n,m} y_m + \sum_{m \in R} p_{n,m} \qquad \textit{for } n \in T.$$

(The solution $(y_n)_{n \in T}$ is minimal when for every non-negative solution $(x_n)_{n \in T}$ the inequalities $y_n \le x_n$ for all $n \in T$ hold.)

Now, since $R = \{0\}$ and $T = \{n | n \ge 1\}$, the desired $(Q_n)_{n \ge 1}$ are the minimal non-negative solution of our system $F(\mathbf{x}) = \mathbf{x}$. We therefore obtain

LEMMA 3.6 (LIMITING VALUE OF Q_n). *We have $\lim_{n \to \infty} Q_n = 0$.*

Proof: We claim that the solution of $F(\mathbf{x}) = \mathbf{x}$, $\mathbf{x} \in \mathbf{l}^\infty(0)$ with $\lim_{i \to \infty} x^{(i)} = 0$ which exists by Lemma 3.4 and by the results of sections 4 and 5 is indeed the minimal solution, i.e., the required sequence $\mathbf{Q} = (Q_n)_{n \ge 1}$: Suppose there exists a non-negative solution $\mathbf{y} \in \mathbf{l}^\infty$ of $F(\mathbf{y}) = \mathbf{y}$ having the property $\mathbf{y} \le \mathbf{x}$. Then, $y^{(i)} \le x^{(i)}$ for all $i \ge 1$ guarantees $\mathbf{y} \in \mathbf{l}^\infty(0)$, and the uniqueness of the solution in $\mathbf{l}^\infty(0)$ by virtue of Lemma 3.4 establishes $\mathbf{y} = \mathbf{x}$. ▮

The following Sections 4 and 5 are devoted to finding sequences $\mathbf{0} \le \mathbf{a} \le \mathbf{b}$ in $\mathbf{l}^\infty(0)$ with $F(\mathbf{a}) \ge \mathbf{a}$ and $F(\mathbf{b}) \le \mathbf{b}$ as required by Lemma 3.4. They actually establish the existence of $(Q_n)_{n \ge 1}$ and form lower and upper bounds, respectively. It is worth mentioning that it is possible to employ a simple iteration to provide tighter bounds: $\mathbf{a} \le F(\mathbf{a}) \le F(\mathbf{Q}) \le F(\mathbf{b}) \le \mathbf{b}$ implies $F(\mathbf{a}) \le \mathbf{Q} \le F(\mathbf{b})$, and $\|F(\mathbf{a}) - F(\mathbf{b})\| < \|\mathbf{a} - \mathbf{b}\|$ according to Lemma 3.1. By the way, note the possibility of the (initial) choice $\mathbf{a} = \mathbf{0}$.

In addition, Lemma 3.4 says that, besides the trivial solution $\mathbf{T} = (T_n)_{n \ge 1}$, $T_n = 1$ for all $n \ge 1$, there exists another solution $\mathbf{Q} = (Q_n)_{n \ge 1}$. This fact, however, suffices to prove that the Markov chain $\{\mathcal{N}_k\}$ is transient, i.e., our Theorem 2.1. We only need to apply the following

CRITERION [**FE1**, P. 402]. *In an irreducible Markov chain with states $\{0, 1, \dots\}$ and transition probabilities $p_{n,m}$, the state 0 is recurrent if, and only if, the linear system*

$$x_n = \sum_{m \ge 1} p_{n,m} x_m \qquad \textit{for } n \ge 1$$

admits no solution with $0 \le x_n \le 1$ except $x_n = 0$ for all n.

Therefore, $\{\mathcal{N}_k\}$ being a recurrent Markov chain would require that $\mathrm{L}\mathbf{x} = \mathbf{x}$ admits only the trivial solution $\mathbf{x} = \mathbf{0}$. However, $\mathbf{T} - \mathbf{Q} \ne \mathbf{0}$ provides another solution, yielding a contradiction and proving Theorem 2.1.

4. AN UPPER BOUND FOR Q_n

This section is devoted to the construction of a convergent sequence $\mathbf{b} = (b_n)_{n \ge 1} \in \mathbf{l}^\infty(0)$ with the properties $\lim_{n \to \infty} b_n = 0$ and $F(\mathbf{b}) \le \mathbf{b}$, forming an upper bound for $(Q_n)_{n \ge 1}$

according to our expositions in Section 3. Consider the following infinite dimensional system of linear equations

$$\begin{aligned} b_1 &= e^{-\lambda}\Big(p + (1-p)(1+\lambda)b_1 + \big(e^{\lambda} - 1 - \lambda(1-p)\big)b_2\Big) \\ b_n &= e^{-\lambda}\Big(np(1-p)^{n-1}b_{n-1} + \big(\lambda(1-p)^n + 1 - np(1-p)^{n-1}\big)b_n \\ &\qquad + \big(e^{\lambda} - 1 - \lambda(1-p)^n\big)b_{n+1}\Big) \quad \text{for } n \geq 2, \end{aligned} \tag{4.1}$$

which is similar to our original one; it corresponds to a modified Markov process where all (former) state transitions from state n to state $n+j$, $j \geq 2$ lead to state $n+1$ instead. Note that the general equation for b_n above is valid for $n = 1$, if we define

$$b_0 = 1. \tag{4.2}$$

In the sequel, we will prove the existence of a solution of (4.1) with $\lim_{n\to\infty} b_n = 0$, which is monotonically decreasing, i.e., $b_{n+j} \leq b_{n+1}$ for $n \geq 1$, $j \geq 1$. Using this property in (4.1), the required validity of $F(\mathbf{b}) \leq \mathbf{b}$ for our solution follows immediately.

Now, a close inspection of system (4.1) reveals

$$(b_n - b_{n+1})(e^{\lambda} - 1 - \lambda(1-p)^n) - (b_{n-1} - b_n)np(1-p)^{n-1} = 0 \qquad \text{for } n \geq 1.$$

Defining

$$q_n = \frac{b_n - b_{n+1}}{b_{n-1} - b_n} = \frac{np(1-p)^{n-1}}{e^{\lambda} - 1 - \lambda(1-p)^n} \qquad \text{for } n \geq 1 \tag{4.3}$$

and

$$\begin{aligned} P_n &= \prod_{i=1}^{n} q_i = \Big(\frac{p}{e^{\lambda}-1}\Big)^n \frac{n!(1-p)^{n(n-1)/2}}{\big(1 - \frac{\lambda}{e^{\lambda}-1}(1-p)\big)\cdots\big(1 - \frac{\lambda}{e^{\lambda}-1}(1-p)^n\big)} \qquad \text{for } n \geq 1, \\ P_0 &= 1, \end{aligned} \tag{4.4}$$

we obtain $b_n - b_{n+1} = (1-b_1)P_n$ for $n \geq 0$ by multiplying equation (4.3) by P_{n-1}, and therefore

$$b_n = 1 - (1-b_1)\sum_{i=0}^{n-1} P_i$$

for $n \geq 1$. That is, every solution of (4.1) depends on the choice of b_1 only. Setting

$$b_1 = 1 - 1/S \qquad \text{where} \qquad S = \sum_{i\geq 0} P_i = 1 + q_1 + q_1 q_2 + \cdots < \infty, \tag{4.5}$$

we have $\lim_{n\to\infty} b_n = 0$. Moreover, $(b_n)_{n\geq 1}$ is monotonically decreasing as required since all $q_n > 0$.

In order to develop an asymptotic expansion for S as $\lambda \to 0$ and $p \to 0$, we need more detailed informations concerning P_n. Actually, we shall establish that the sum for S has a dominating term P_m, $1 < m < \infty$.

In order to show that the denominator of P_m provides (relatively) small contributions, we shall need the following

Lemma 4.1 (Asymptotics of an Infinite Product). *We have*

$$\log \prod_{n\geq 1}\left(1-(1-p)^n\right) = -\frac{\pi^2}{6p} - \frac{\log p}{2} + \frac{\log 2\pi}{2} + O(p) \qquad \textit{for } p \to 0.$$

Proof: See Appendix. ∎

By virtue of this result, it is not hard to prove that the denominator of (4.4) obeys the following bounds:

Lemma 4.2 (Bounds for the Denominator of P_n). *The denominator of P_n, $n \geq 1$*

$$D_n = \prod_{i=1}^{n}(1 - \frac{\lambda}{e^\lambda - 1}(1-p)^i)$$

fulfills

$$1 < D_n^{-1} = \exp\Big(O(\frac{1}{-\log(1-p)})\Big) \qquad \textit{for } p \to 0.$$

Proof: Noting that

$$\prod_{i\geq 1}(1-(1-p)^i) < \prod_{i=1}^{n}(1 - \frac{\lambda}{e^\lambda - 1}(1-p)^i) < 1$$

since $\lambda < e^\lambda - 1$, the lower bound for D_n^{-1} follows immediately; the upper bound is a consequence of a very weak form of the result of Lemma 4.1. ∎

Next, we concern ourselves with the numerator of (4.4). We prove the following

Lemma 4.3 (Maximum of the Numerator of P_n). *The numerator of P_n, $n \geq 1$*

$$F_n = \left(\frac{p}{e^\lambda - 1}\right)^n n!(1-p)^{n(n-1)/2} = \frac{np(1-p)^{n-1}}{e^\lambda - 1}F_{n-1}$$

attains its maximum value F_m at

$$m = \frac{\log \lambda^{-1} + \log\log \lambda^{-1} + O(1)}{-\log(1-p)}$$

and

$$\log F_m = \frac{\left(\log \lambda^{-1}\right)^2 + 2\log \lambda^{-1} \log\log \lambda^{-1} + O(\log \lambda^{-1})}{-2\log(1-p)}.$$

Proof: Expanding the logarithm of the positive function

$$h(y) = \frac{yp(1-p)^{y-1}}{e^\lambda - 1}$$

by using the straightforward expansions

$$\begin{aligned} \log \frac{1}{e^{\lambda}-1} &= \log \lambda^{-1} + O(\lambda) && \text{for } \lambda \to 0, \\ -\log(1-p) &= p\big(1+O(p)\big) && \text{for } p \to 0, \\ \log\big(-\log(1-p)\big) &= \log p + O(p) && \text{for } p \to 0, \end{aligned} \tag{4.6}$$

we obtain

$$\log h(y) = y\log(1-p) + \log y + \log \lambda^{-1} + \log p + O(p) + O(\lambda). \tag{4.7}$$

for $\lambda \to 0$, $p \to 0$. Hence it follows that there is a single zero y_0 of $\log h(y)$ and

$$y_0 \le y_1 = \frac{2\log \lambda^{-1}}{-\log(1-p)}, \tag{4.8}$$

since it is easily checked that $\log h(y_1) < 0$ (provided that λ and/or p is sufficiently small, of course). Repeated bootstrapping yields an asymptotic expression of y_0 for $\lambda \to 0$, $p \to 0$, namely

$$y_0 = \frac{\log \lambda^{-1} + \log\log \lambda^{-1} + O\left(\frac{\log\log \lambda^{-1}}{\log \lambda^{-1}}\right) + O(p) + O(\lambda)}{-\log(1-p)}.$$

Now, since $h(y) > 1$ for $y < y_0$ and $h(y) < 1$ for $y > y_0$, it is clear that F_n attains its maximum at $n = m = \lfloor y_0 \rfloor = y_0 + O(1)$. Finally, the desired asymptotic expression for $\log F_m$ is easily derived by using Stirling's formula

$$\log m! = \log \Gamma(m+1) = m\log m - m + \frac{\log m}{2} + O(1) \qquad \text{for } m \to \infty.$$

∎

Putting everything together, we are able to show

LEMMA 4.4 (ASYMPTOTIC EXPANSION OF S). *For $\lambda \to 0$ and $p \to 0$, we have*

$$\log S = \frac{\left(\log \lambda^{-1}\right)^2 + 2\log \lambda^{-1} \log\log \lambda^{-1} + O(\log \lambda^{-1})}{-2\log(1-p)}.$$

Proof: Remembering the proof of Lemma 4.3, especially equation (4.8), we obviously have

$$P_m \le S = \sum_{n\ge 0} P_n \le \frac{2\log \lambda^{-1}}{-\log(1-p)} P_m + O(1).$$

Taking logarithms and applying the result of Lemma 4.3, our result follows. ∎

Now we easily obtain

THEOREM 4.5 (ASYMPTOTIC UPPER BOUND). *We have the following asymptotic upper bound:*

$$\frac{1}{1-B} \le \exp\Big(\frac{(\log\lambda^{-1})^2 + 2\log\lambda^{-1}\log\log\lambda^{-1} + O(\log\lambda^{-1})}{-2\log(1-p)}\Big).$$

Proof: Remembering equation (2.10), we obtain

$$\begin{aligned}1-B &= e^{-\lambda}\sum_{j\ge 2}\frac{\lambda^j}{j!}(1-Q_j) \ge e^{-\lambda}\sum_{j\ge 2}\frac{\lambda^j}{j!}(1-b_j)\\ &> e^{-\lambda}(e^{\lambda}-\lambda-1)(1-b_2)\\ &= \big(1-e^{-\lambda}(1+\lambda)\big)(1+q_1)(1-b_1),\end{aligned}$$

cf. definition (4.3). Applying definition (4.5) and Lemma 4.4, our result follows. ∎

5. A LOWER BOUND FOR Q_n

The purpose of this section is the construction of a sequence $\mathbf{a} = (a_n)_{n\ge 1} \in \mathbf{l}^\infty(0)$ with $F(\mathbf{a}) \ge \mathbf{a}$ and $\mathbf{a} \le \mathbf{b}$, forming a lower bound for our desired sequence $(Q_n)_{n\ge 1}$ according to Section 3. Consider the following infinite system of linear equations

$$\begin{aligned}a_1 &= e^{-\phi_1}\Big(p + (1-p)(1+\phi_1)a_1 + \big(e^{\phi_1} - 1 - \phi_1(1-p)\big)a_2\Big)\\ a_n &= e^{-\phi_n}\Big(np(1-p)^{n-1}a_{n-1} + \big(\phi_n(1-p)^n + 1 - np(1-p)^{n-1}\big)a_n \\ &\qquad + \big(e^{\phi_n} - 1 - \phi_n(1-p)^n\big)a_{n+1}\Big) \qquad \text{for } n \ge 2,\end{aligned} \tag{5.1}$$

which is very similar to the (upper bound) system (4.1) of Section 4 and has an obvious Markov chain interpretation too. Note that the general equation is valid for $n \ge 1$, if we define $a_0 = 1$ by convention.

Now, the "lucky" replacement of λ by ϕ_n *depending on* $n \ge 1$ allows the construction of a lower bound, too, by means of choosing a suitable "large" sequence $\mathbf{\Phi} = (\phi_n)_{n\ge 1}$. Actually, we shall show that system (5.1) has a unique solution $\mathbf{a} \in \mathbf{l}^\infty(0)$ satisfying the conditions mentioned at the beginning of this section, if

$$\phi_n = \begin{cases} C_1\lambda^{2/3}, & \text{for } n \le 1/p \\ C_2\lambda, & \text{for } n > 1/p \end{cases} \tag{5.2}$$

with $\phi_n \ge \lambda$ for all $n \ge 1$ and C_1, C_2 fixed accordingly in the sequel.

First, it is clear that our considerations from Section 4 carry over almost literally. We easily obtain

$$a_n = 1 - (1-a_1)\sum_{i=0}^{n-1}\overline{P}_i$$

for $n \ge 1$, where

$$\overline{P}_n = \prod_{i=1}^{n}\overline{q}_i$$

and

$$\overline{q}_n = \frac{a_n - a_{n+1}}{a_{n-1} - a_n} = \frac{np(1-p)^{n-1}}{e^{\phi_n} - 1 - \phi_n(1-p)^n}.$$

Again, the solution depends on the choice of a_1 only. Setting

$$a_1 = 1 - 1/\overline{S} \qquad \text{where} \qquad \overline{S} = \sum_{i\geq 0} \overline{P}_i = 1 + \overline{q}_1 + \overline{q}_1\overline{q}_2 + \cdots < \infty, \tag{5.3}$$

we obtain $\lim_{n\to\infty} a_n = 0$, and $(a_n)_{n\geq 1}$ is monotonically decreasing since all $\overline{q}_n > 0$. Moreover, we have the following coarse estimation

LEMMA 5.1 (ESTIMATION OF $\overline{q}_n$). *For all $n \geq 1$, we have*

$$\overline{q}_n = \frac{np(1-p)^{n-1}}{e^{\phi_n} - 1 - \phi_n(1-p)^n} \leq \frac{1}{\phi_n} \leq \frac{1}{\lambda}.$$

Proof: Using $1 = (1-p+p)^n \geq (1-p)^n + np(1-p)^{n-1}$ and $x < e^x - 1$ for any $x > 0$, we find

$$1 \geq \frac{x}{e^x - 1}(1-p)^n + \frac{x}{e^x - 1} np(1-p)^{n-1}$$

and therefore

$$\overline{q}_n = \frac{np(1-p)^{n-1}}{e^{\phi_n} - 1 - \phi_n(1-p)^n} \leq \frac{1}{\phi_n} \leq \frac{1}{\lambda},$$

since $\phi_n \geq \lambda$ according to (5.2). ∎

We find it convenient to rewrite our system (5.1) as a mapping similar to Section 3, i.e.,

$$G_{\mathbf{\Phi}} : \mathbf{l}^\infty \mapsto \mathbf{l}^\infty, \quad \mathbf{x} \mapsto \mathrm{M}_{\mathbf{\Phi}}\mathbf{x} + \mathbf{c}_{\mathbf{\Phi}}, \tag{5.4}$$

where

$$\begin{aligned} (\mathrm{M}_{\mathbf{\Phi}}\mathbf{x})^{(1)} &= e^{-\phi_1}\Big((1-p)(1+\phi_1)x_1 + (e^{\phi_1} - 1 - \phi_1(1-p))x_2\Big) \\ (\mathrm{M}_{\mathbf{\Phi}}\mathbf{x})^{(n)} &= e^{-\phi_n}\Big(np(1-p)^{n-1}x_{n-1} + (\phi_n(1-p)^n + 1 - np(1-p)^{n-1})x_n \\ &\qquad + (e^{\phi_n} - 1 - \phi_n(1-p)^n)x_{n+1}\Big) \quad \text{for } n \geq 2, \end{aligned} \tag{5.5}$$

and

$$c_{\mathbf{\Phi}}^{(n)} = \begin{cases} pe^{-\phi_1}, & \text{for } n = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{5.6}$$

In order to establish that $F(\mathbf{a}) \geq \mathbf{a}$, we consider the difference

$$a_n - (F(\mathbf{a}))^{(n)} = \Big[(G_{\mathbf{\Phi}}(\mathbf{a}))^{(n)} - (G_{\mathbf{\Lambda}}(\mathbf{a}))^{(n)}\Big] + \Big[(G_{\mathbf{\Lambda}}(\mathbf{a}))^{(n)} - (F(\mathbf{a}))^{(n)}\Big], \tag{5.7}$$

remember that $\mathbf{a}$ is the unique fixed point of $G_{\mathbf{\Phi}}$. The sequence $\mathbf{\Lambda}$ denotes the sequence with all elements equal to λ. Thus, $G_{\mathbf{\Lambda}}$ coincides with the upper bound scenario in Section 4 and has the unique fixed point $\mathbf{b} = (b_n)_{n\geq 1} \in \mathbf{l}^\infty(0)$.

We start working on the second term above by the following

LEMMA 5.2. *Let Φ be a real sequence with $\phi_n \geq \lambda$ for $n \geq 1$, and $\mathbf{a} \in \mathbf{l}^\infty(0)$ the unique fixed point of G_Φ. Then we have*

$$\frac{e^{-\lambda}}{2}\lambda^2(a_{n+1} - a_{n+2}) < \left(G_\Lambda(\mathbf{a})\right)^{(n)} - \left(F(\mathbf{a})\right)^{(n)} < \frac{e}{2}\lambda^2(a_{n+1} - a_{n+2}).$$

Proof: See Appendix. ▮

Now, in order to establish that $F(\mathbf{a}) \geq \mathbf{a}$, we only have to show that our special choice (5.2) ensures that

$$\left(G_\Phi(\mathbf{a})\right)^{(n)} - \left(G_\Lambda(\mathbf{a})\right)^{(n)} \leq -\frac{e}{2}\lambda^2(a_{n+1} - a_{n+2}) \tag{5.8}$$

for all $n \geq 1$, according to equation (5.7) and Lemma 5.2. We need the following preperational lemma:

LEMMA 5.3. *We have*

$$\left(G_\Lambda(\mathbf{a})\right)^{(n)} - \left(G_\Phi(\mathbf{a})\right)^{(n)} \geq (a_n - a_{n+1})(\phi_n - \lambda)\left(1 - e^{-\phi_n}(1-p)^n - \frac{\phi_n}{2}\right)$$

Proof: See Appendix. ▮

Since $a_{n+1} - a_{n+2} = \overline{q}_{n+1}(a_n - a_{n+1})$ and $\overline{q}_{n+1} \leq 1/\phi_{n+1}$ according to Lemma 5.1, it remains to show that

$$\phi_{n+1}(\phi_n - \lambda)\left(1 - e^{-\phi_n}(1-p)^n - \frac{\phi_n}{2}\right) \geq \frac{e}{2}\lambda^2 \tag{5.9}$$

in order to justify equation (5.8). We have to distinguish 2 cases, namely

(1) For $n \leq 1/p - 1$,

$$1 - e^{-\phi_n}(1-p)^n - \frac{\phi_n}{2} \geq 1 - e^{-\phi_n} - \frac{\phi_n}{2} \geq \frac{\phi_n}{2} - \frac{\phi_n^2}{2}$$

reveals that there is a constant $C_1 > 1$ such that inequality (5.9) is satisfied for $\phi_n = \phi_{n+1} = C_1\lambda^{2/3}$, provided λ is small enough and $p > 0$, of course.

(2) For $n > 1/p$, it is possible to sharpen the bound

$$1 - e^{-\phi_n}(1-p)^n - \frac{\phi_n}{2} \geq 1 - (1-p)^n - \frac{\phi_n}{2} \geq C > 0.$$

Hence, there exists a constant $C_2 > 1$ such that inequality (5.9) is justified for $\phi_n = \phi_{n+1} = C_2\lambda$, provided λ is small enough and $p > 0$, too.

The "mixed case" $\phi_n = C_1\lambda^{2/3}$ and $\phi_{n+1} = C_2\lambda$ may be handeled by choosing C_2 large enough.

Thus, we have proved that our sequence $\mathbf{a}$ satisfies $F(\mathbf{a}) \geq \mathbf{a}$. In order to establish that $\mathbf{a} \leq \mathbf{b}$, it suffices to show that $G_{\Phi}(\mathbf{b}) \leq \mathbf{b}$. This follows from a pendant of our Lemma 3.4, this time applicable to G_{Φ} (the proof runs along the very same way). It says that, if $G_{\Phi}(\mathbf{b}) \leq \mathbf{b}$, there exists a unique fixed point $\mathbf{a}^* \in \mathbf{l}^{\infty}(0)$ with $\mathbf{0} \leq \mathbf{a}^* \leq \mathbf{b}$. Now, since $\mathbf{a} \in \mathbf{l}^{\infty}(0)$ is a fixed point of G_{Φ}, it is clear that $\mathbf{a} = \mathbf{a}^* \leq \mathbf{b}$.

In order to show that $G_{\Phi}(\mathbf{b}) \leq \mathbf{b}$, we use

$$h_n(\mu) = e^{-\mu}\Big(np(1-p)^{n-1}b_{n-1} + \big(\mu(1-p)^n + 1 - np(1-p)^{n-1}\big)b_n + \big(e^{\mu} - 1 - \mu(1-p)^n\big)b_{n+1}\Big).$$

Then, we have $h_n(\phi_n) = \big(G_{\Phi}(\mathbf{b})\big)^{(n)}$ and $h_n(\lambda) = b_n$. Now

$$h_n'(\mu) = -e^{-\mu}\Big(np(1-p)^{n-1}(b_{n-1} - b_n) + \big(1 - (1-p)^n + \mu(1-p)^n\big)(b_n - b_{n+1})\Big) < 0,$$

and $\phi_n \geq \lambda$ imply

$$b_n = h_n(\lambda) \geq h_n(\phi_n) = \big(G_{\Phi}(\mathbf{b})\big)^{(n)}$$

and we are done.

At last, we complete our derivations by providing the necessary asymptotics. However, since ϕ_n is close to λ according to (5.2), we may use the proofs of Lemma 4.3 and 4.3 almost literally to establish

LEMMA 5.4 (ASYMPTOTIC EXPANSION OF $\overline{S}$). *For $\lambda \to 0$ and $p \to 0$, we have*

$$\log \overline{S} = \frac{(\log \lambda^{-1})^2 + 2\log\lambda^{-1}\log\log\lambda^{-1} + O(\log\lambda^{-1})}{-2\log(1-p)}.$$

Now it is easy to get

THEOREM 5.5 (ASYMPTOTIC LOWER BOUND). *We have the following asymptotic lower bound*

$$\frac{1}{1-B} \geq \exp\Big(\frac{(\log \lambda^{-1})^2 + 2\log\lambda^{-1}\log\log\lambda^{-1} + O(\log\lambda^{-1})}{-2\log(1-p)}\Big).$$

Proof: Remembering equation (2.10) and Lemma 5.1, we find

$$\begin{aligned}
1 - B &= e^{-\lambda}\sum_{j\geq 2}\frac{\lambda^j}{j!}(1 - Q_j) \\
&\leq e^{-\lambda}\sum_{j\geq 2}\frac{\lambda^j}{j!}(1 - a_j) = e^{-\lambda}(1-a_1)\sum_{j\geq 2}\frac{\lambda^j}{j!}\sum_{k=0}^{j-1}\overline{P}_k \\
&= (1-a_1)(1 - e^{-\lambda}(1+\lambda)) + e^{-\lambda}(1-a_1)\sum_{k\geq 2}\overline{P}_{k-1}\sum_{j\geq k}\frac{\lambda^j}{j!} \\
&\leq (1-a_1)\lambda^2 + e^{-\lambda}(1-a_1)\lambda\sum_{k\geq 2}\frac{1}{k!}\sum_{j\geq 0}\frac{k!}{(j+k)!}\lambda^j \\
&\leq (1-a_1)\lambda C
\end{aligned}$$

for some constant C. Applying definition (5.3) and Lemma 5.4, our result follows. ∎

6. Bounds for Q'_n

In this section, we shall establish an upper and a lower bound for $B' = B'(1)$. We start with the derivation of the upper bound, which is much more involved than the lower one, and claim

THEOREM 6.1. *There is a constant $C > 0$ such that*

$$B' \leq \exp\Big(\frac{C}{p}\log\lambda^{-1}\Big) \tag{6.1}$$

uniformly for $\lambda \to 0$, $p \to 0$.

By (2.4), the generating functions $Q'_n(z)$ satisfy the functional equations

$$\begin{aligned} Q'_1(z) &= \frac{1}{z}Q_1(z) + ze^{-\lambda}\Big((1-p)(1+\lambda)Q'_1(z) + \lambda p Q'_2(z) + \sum_{j\geq 2}\frac{\lambda^j}{j!}Q'_{j+1}(z)\Big) \\ Q'_n(z) &= \frac{1}{z}Q_n(z) + ze^{-\lambda}\Big(np(1-p)^{n-1}Q'_{n-1}(z) + (1-np(1-p)^{n-1} + \lambda(1-p)^n)Q'_n(z) \\ &\quad + \lambda(1-(1-p)^n)Q'_{n+1}(z) + \sum_{j\geq 2}\frac{\lambda^j}{j!}Q'_{n+j}(z)\Big) \quad \text{for } n \geq 2. \end{aligned} \tag{6.2}$$

Setting $\mathbf{Q}(z) = \big(Q_n(z)\big)_{n\geq 1}$ and $\mathbf{Q}'(z) = \big(Q'_n(z)\big)_{n\geq 1}$, we may rewrite system (6.2) as a mapping similar to $\mathbf{Q}(z) = z(\mathrm{L}\mathbf{Q}(z) + \mathbf{c})$, namely

$$\mathbf{Q}'(z) = \frac{1}{z}\mathbf{Q}(z) + z\mathrm{L}\mathbf{Q}'(z). \tag{6.3}$$

Having solved this fixed point problem for some specific $z \leq 1$, $B'(z)$ may be evaluated via

$$B'(z) = \frac{1}{z}B(z) + e^{-\lambda}\sum_{j\geq 2}\frac{\lambda^j}{j!}Q'_j(z). \tag{6.4}$$

Thus, setting $z = 1$ we get a problem quite similar to the already solved one, with the only (but essential!) difference that $Q'_n = Q'_n(1) = 1$, $n \geq 1$ is no solution. Nevertheless, the basic properties stated in the following lemma are almost the same as before: Using our well-established notation $\mathbf{Q} = (Q_n)_{n\geq 1}$ and $\mathbf{Q}' = (Q'_n)_{n\geq 1}$, we have

LEMMA 6.2 (AN UPPER BOUND FOR Q'_n). *Suppose that there is a non-negative sequence $\mathbf{c} = (c_n)_{n\geq 1} \in \mathbf{l}^\infty(0)$ with $\mathbf{c} \geq \mathbf{Q} + \mathrm{L}\mathbf{c}$. Then, $\mathbf{Q}'$ is the unique solution of $\mathbf{x} = \mathbf{Q} + \mathrm{L}\mathbf{x}$ under the restriction $\mathbf{x} \in \mathbf{l}^\infty(0)$ and $\mathbf{0} \leq \mathbf{Q}' \leq \mathbf{c}$.*

Proof: Since $\frac{1}{z}\mathbf{Q}(z) \leq \mathbf{Q}(1)$ for $0 < z \leq 1$ we conclude by Lemma 3.4 that $\mathbf{a} = \mathbf{0} \leq \mathbf{b} = \mathbf{c}$ are bounds for the unique solution $\mathbf{x}$ of $\mathbf{x} = \frac{1}{z}\mathbf{Q}(z) + z\mathrm{L}\mathbf{x}$ under the restriction $\mathbf{x} \in \mathbf{l}^\infty(0)$. But, this fixed point problem has exactly one solution for $|z| < 1$ by Banach's fixed point theorem, namely $\mathbf{Q}'(z)$. So we have $\mathbf{0} \leq \mathbf{Q}'(z) \leq \mathbf{c}$ for $0 < z < 1$, which implies that $\mathbf{Q}'(1)$ exists and is bounded by $\mathbf{0} \leq \mathbf{Q}'(1) \leq \mathbf{c}$. ∎

Therefore, the remaining problem is to find such a sequence $\mathbf{c}$. Our basic idea is to look for certain partial solutions first, and to put them together in order to construct a global solution $\mathbf{c}$ satisfying $\mathbf{c} \geq \mathbf{Q} + \mathbf{L}\mathbf{c}$. Keep in mind that, if $\mathbf{c} = (c_n)_{n\geq 1}$ denotes a sequence with $\mathbf{c} - \mathbf{L}\mathbf{c} \geq \mathbf{Q}$, any sequence $\mathbf{d} = (Kc_n)_{n\geq 1}$ fulfills $\mathbf{d} - \mathbf{L}\mathbf{d} \geq \mathbf{Q}$, too, provided that $K \geq 1$.

Denoting the nearest integer to $(\log \lambda^{-1})/(-\log(1-p))$ by n_0 (which causes $(1-p)^{n_0-1} > \lambda$ and $(1-p)^{n_0+1} < \lambda$), it is not hard to establish

LEMMA 6.3 (APPROXIMATION FOR LARGE n). *There exists a constant $C > 0$ such that $\mathbf{c} = (c_n)_{n\geq 1}$, where*

$$c_n = b_{n-1} \exp\Big(\frac{C}{p} \log \lambda^{-1}\Big)$$

with b_n denoting the solution of the (upper bound) system (4.1) satisfies

$$c_n - (\mathbf{L}\mathbf{c})^{(n)} \geq b_n \geq Q_n$$

for $n \geq n_0$.

Proof: See Appendix. ∎

Next, we have

LEMMA 6.4 (APPROXIMATION FOR MEDIUM n). *There exist some constants $C_1 > 0$, $C_2 > 0$ such that $\mathbf{c} = (c_n)_{n\geq 1}$, where*

$$c_n = \frac{C_1}{\lambda^2} 2^n$$

satisfies

$$c_n - (\mathbf{L}\mathbf{c})^{(n)} \geq 1 \geq Q_n$$

for $\max(1, C_2\lambda^2/p) \leq n \leq n_0$.

Proof: A simple computation shows

$$\begin{aligned} c_n - (\mathbf{L}\mathbf{c})^{(n)} &= \frac{C_1}{\lambda^2} 2^n e^{-\lambda}\Big(\frac{1}{2} np(1-p)^{n-1} + \lambda(1-p)^n - e^{2\lambda} + e^{\lambda}\Big) \\ &\geq \frac{C_1}{\lambda^2} 2^n C_3 \lambda^2 \geq C_1 C_3 \geq 1 \end{aligned}$$

for $\max(1, C_2\lambda^2/p) \leq n \leq n_0$. ∎

Lemmata 6.3 and 6.4 permit the construction of a sequence $\mathbf{c}$ with $\mathbf{c} \geq \mathbf{Q} + \mathbf{L}\mathbf{c}$ as required by Theorem 6.1 in the case $p \geq C_2\lambda^2$: By Lemma 6.3 there exists a constant $C > 0$ such that $\mathbf{c}' = (c'_n)_{n\geq 1}$ with

$$c'_n = b_{n-1} \exp\Big(\frac{C}{p} \log \lambda^{-1}\Big)$$

satisfies $c_n' - (\mathbf{L}\mathbf{c}')^{(n)} \geq Q_n$ for $n \geq n_0$, and by Lemma 6.4 there exists a constant $C_1 > 0$ such that

$$c_n'' = \frac{C_1}{\lambda^2} 2^n$$

satisfies $c_n'' - (\mathbf{L}\mathbf{c}'')^{(n)} \geq Q_n$ for $n \leq n_0$. Now, if

$$\frac{C_1}{\lambda^2} 2^{n_0} < b_{n_0-1} \exp\Big(\frac{C}{p} \log \lambda^{-1}\Big),$$

set $C_1' = \lambda^2 2^{-n_0} b_{n_0-1} \exp\Big(\frac{C}{p} \log \lambda^{-1}\Big)$ and

$$c_n = \frac{C_1'}{\lambda^2} 2^n \qquad \text{for } n \leq n_0$$

$$c_n = b_{n-1} \exp\Big(\frac{C}{p} \log \lambda^{-1}\Big) \qquad \text{for } n \leq n_0.$$

If

$$\frac{C_1}{\lambda^2} 2^{n_0} \geq b_{n_0-1} \exp\Big(\frac{C}{p} \log \lambda^{-1}\Big),$$

set $C_1'' = C_1 \lambda^{-2} 2^{n_0} \exp\Big(-\frac{C}{p} \log \lambda^{-1}\Big)/b_{n_0-1}$ and

$$c_n = \frac{C_1}{\lambda^2} 2^n \qquad \text{for } n \leq n_0$$

$$c_n = C_1'' b_{n-1} \exp\Big(\frac{C}{p} \log \lambda^{-1}\Big) \qquad \text{for } n \leq n_0.$$

This sequence $\mathbf{c} = (c_n)_{n \geq 1}$ obviously satisfies $\mathbf{c} \geq \mathbf{Q} + \mathbf{L}\mathbf{c}$. Therefore, using a coarse estimation of (2.11), we find

$$B' \leq 1 + \max_{n \geq 2} c_n \leq \exp\Big(\frac{C'}{p} \log \lambda^{-1}\Big) \tag{6.5}$$

for some constant $C' > 0$, which establishes Theorem 6.1 in the case of $p \geq C_2 \lambda^2$.

Unfortunately, the case $p < C_2\lambda^2$ introduces unpleasant difficulties. This fact becomes intuitively clear when the drift D_n is considered: According to our remarks following equation (1.2), there is some n_1 such that $D_n < 0$ for $n \leq n_1$ and $D_n \geq 0$ for $n > n_1$. However, the range where the drift D_n is negative starts with $n_0 = 1$ only if $p \geq C_2\lambda^2$. For $p \leq C_2\lambda^2$, there exists some $1 < n_0 < n_1$ such that $D_n \geq 0$ for $n < n_0$, $D_n < 0$ for $n_0 \leq n \leq n_1$, and $D_n \geq 0$ for $n > n_1$. This means that an ALOHA system with backlog between n_0 and n_1 tends to decrease the backlog towards n_0, but is very reluctant to pass the gap between n_0 and 0; an actual collision resolution regarding a small backlog needs more time if p is decreased, since the number of idle slots increases.

Nevertheless, the asymptotic expansion for the probability B that a busy period is finite is uniform for $\lambda \to 0$, $p \to 0$, i.e., gets larger if p and/or λ are reduced. Note that $B/(1-B)$ is the expected number of busy periods and B'/B is the expected length of a busy period.

Now, using some technical refinements it is possible to prove

LEMMA 6.5 (APPROXIMATION FOR SMALL n AND SMALL p). *Let n_1 be the nearest integer to $C_2\lambda^2/p$ and suppose that $n_1 > 1$. Then there exist constants $C_3 > 0$, $C_4 > C_2$ such that $\mathbf{c} = (c_n)_{n\geq 1}$ with*

$$c_n = \sum_{k=n_1-n+1}^{n_1} a_k \tag{6.6}$$

for $n \leq n_1$, where

$$a_k = \exp\Big(C_3 \log \lambda^{-1}\Big) \frac{(n_1-k)!}{n_1!} \Big(\frac{C_4\lambda^2}{p}\Big)^k \tag{6.7}$$

and

$$c_n = a_0 2^{n-n_1} + c_{n_1-1} \tag{6.8}$$

for $n > n_1$ satisfies

$$c_n - (\mathrm{L}\mathbf{c})^{(n)} \geq 1 \geq Q_n$$

for $n < n_1$.

Proof: See Appendix. ∎

Using this additional lemma, it is not hard to construct a global sequence $\mathbf{c} = (c_n)_{n\geq 1}$ satisfying $\mathbf{c} \geq \mathbf{Q} + \mathrm{L}\mathbf{c}$ for $p \leq C_2\lambda^2$, too. This construction runs along our considerations preceeding equation (6.5); however, we suppress details for the sake of shorteness. Now, by virtue of

$$\frac{1}{n_1^k} \leq \frac{(n_1-k)!}{n_1!} \leq \frac{e^k}{n_1^k},$$

we easily obtain the coarse estimation

$$\max_{1\leq n\leq n_1} c_n \leq \exp\Big(\frac{C''}{p} \log \lambda^{-1}\Big)$$

for some suitable constant $C'' > 0$ (by summing up definition (6.7) for $1 \leq k \leq n_1$). This result combined with inequality (6.5) completes the proof of Theorem 6.1.

What remains to do is to collect our results: Remembering (2.9), it is clear that Theorem 4.5 in conjunction with Theorem 6.1, and Theorem 5.5 together with

$$B' = \sum_{j\geq 1} j b_j \geq \sum_{j\geq 1} b_j = B \geq 1/2,$$

uniformly for $\lambda \to 0$, $p \to 0$, establish our major result as stated in Theorem 2.2 of Section 2: both the resulting upper and the lower bound for $\mathrm{E}[\mathcal{S}]$ are of the same order.

7. CONCLUSIONS

In the previous sections, we computed the average time until destabilization of the slotted ALOHA collision resolution algorithm occurs. Relying on a specific renewal approach based on busy periods (starting from backlog 0 and returning to backlog 0 again), we found that the whole period of successful operation $\mathcal{S}$ consists of a finite number of finite busy periods

$\mathcal{B}$ terminated by an infinite one, and derived a uniform asymptotic expression for E[$\mathcal{S}$] as $\lambda \to 0$ and $p \to 0$. As a byproduct, we obtained a simple proof of the (already known) fact that the Markov chain for slotted ALOHA is transient.

Our actual computations use a pair of upper and lower bounds, which are shown to be asymptotically equivalent up to a factor(!) of order $\exp(O(\log \lambda^{-1}/p))$, which is relatively large. However, the asymptotic evaluation is already somewhat technical (and long enough), and establishing more accurate expressions would involve much more sophisticated techniques, e.g., multivariate Mellin transform techniques. Besides, it is questionable if our bounds are still asymptotically equivalent if lower order terms are considered.

It is clear that such results for small λ and p are (despite of uniformity, cf. the footnote at the end of Section 2) a bit unrealistic from an engineering perspective since, for small λ, one usually chooses a relatively large p. Nevertheless, numerical results for large values may be obtained by a simple numerical iteration of $\mathbf{Q}^{n+1} = F(\mathbf{Q}^n)$ starting from $Q_n^0 = 0$, $n \geq 1$ (and similar for $\mathbf{Q}'^{n+1}$ with $Q_n'^0 = 0$). It produces gradually improved, monotone approximations of Q_n and Q'_n; cf. our remarks following Lemma 3.6.

It is naturally interesting to compare our E[$\mathcal{S}$] with the large deviation result E[$\mathcal{L}$] derived in [GW]. As already mentioned in Section 1, it is possible to extend the latter analysis in order to obtain an asymptotic expression of E[$\mathcal{L}$] as $p \to 0$ and $\lambda \to 0$, cf. equation (1.3). A sketch of the derivations looks as follows: Using the "small-p" approximation of the drift $D_n \approx \lambda - (\lambda + np)e^{-(\lambda+np)}$ and substituting $y = np$, we define the drift function $d(y) = \lambda - (\lambda + y)e^{-(\lambda+y)}$, which is presumed to have two roots y_0 and y_1; we suppress the obvious dependence from λ for notational convenience. According to the large deviation results, the required quantity $c = c(\lambda)$ evaluates to

$$c = \int_{y_0}^{y_1} \Theta^*(y)\, dy$$

where $\Theta^*(y) = \Theta^*(y, \lambda)$ is defined by the following equation concerning the moment generating function $M(\Theta, y)$ of the "small-p" approximations of our Q_n, cf. equation (2.3):

$$M(\Theta^*(y), y) = e^{\lambda(e^{\Theta^*(y)}-1)} - (e^{\Theta^*(y)} - 1)\lambda e^{-(\lambda+y)} + (e^{-\Theta^*(y)} - 1)ye^{-(\lambda+y)} = 1$$

First, is is not hard to see that asymptotically, for small λ, $y_0 \approx 0$ and $y_1 \approx \log\lambda^{-1} + \log\log\lambda^{-1}$. In addition, for y becoming large, we find

$$f(y) = \lambda e^{\Theta^*(y)} \approx ye^{-y}.$$

After a straightforward evaluation, the integral above yields

$$c \approx \frac{1}{2}\log^2\lambda^{-1} + \log\lambda^{-1}\log\log\lambda^{-1}.$$

Therefore, we see that the major terms of E[$\mathcal{L}$] of [**GW**] and our result E[$\mathcal{S}$] are equivalent, though we deal with apparently different models. It seems that the differences are in fact not essential and that we provided another —completely proven— approach to the same result; for example, there is a obvious coincidence between our terminating (i.e., infinite) busy period and the almost deterministic path out of the safe region as predicted by large deviation methods.

It is clear that our approach provides a method of attacking certain related problems, too. For example, a forthcoming paper shall establish that S is approximately exponentially distributed with parameter $\mu = 1/\mathrm{E}[S]$ (as already predicted by large deviation methods, too). In addition, other interesting quantities arising in this context might be investigated. For instance, it is easily seen that $B/(1-B)$ is the expected number of busy periods until destabilization, and that B'/B is the expected length of a finite (that is, a terminating) busy period. Another challenging problem is the development of an asymptotic expression for the average CRI-length Q'_n, which is the average number of slots necessary for resolving an initial collision of n stations, cf. our remarks preceeding definition (2.1).

Finally, we should note that we successfully employed our approach in a very different context, namely queueing systems with deadlines. Due to the general nature of the results obtained (cf. [**DS**]) we presume that certain problems in different fields might be tackled, too. On the other hand, we readily confess that it is questionable if our approach will be successful in conjunction with multi-dimensional Markov chains arising in the analysis of more complex collision resolution algorithms like binary exponential backoff (employed for Ethernet, cf. [**A**]): Our general renewal technique naturally applies, but establishing soluble upper and lower bounds is actually much more complicated than shown in this paper.

Appendix

Proof of Lemma 4.1: Defining

$$f(t) = \sum_{n\geq 1} \log(1-e^{-nt}) = -\sum_{k\geq 1}\frac{1}{k}\sum_{n\geq 1} e^{-nkt},$$

we have

$$\log \prod_{n\geq 1}\left(1-(1-p)^n\right) = f(-\log(1-p)). \tag{A.1}$$

The asymptotic expansion of $f(t)$ as $t \to 0+$ is easily developed by means of Mellin transform techniques, cf. [**VF**] for an overview.

Using the most useful "harmonic" property $\mathcal{M}\big(f(ct);s\big) = c^{-s}\mathcal{M}\big(f(t);s\big)$, the Mellin transform $f^*(s) = \mathcal{M}\big(f(t);s\big)$ of $f(t)$ evaluates to

$$f^*(s) = \int_0^\infty t^{s-1} f(t)\, dt = -\zeta(s+1)\zeta(s)\Gamma(s),$$

which is analytic within the fundamental strip $\Re(s) > 1$. Using the well-known inversion theorem of the Mellin transform, we have

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s) t^{-s}\, ds,$$

where $c > 1$ denotes an arbitrary real constant within the fundamental strip. An asymptotic expansion of $f(t)$ for $t \to 0+$ may be obtained by extending the contour to a closed rectangle (beyond the left of the fundamental strip) and taking into account the residues of the enclosed singularities. We therefore obtain

$$\begin{aligned} f(t) = \operatorname{res}\big(f^*(s)t^{-s}; s = 1\big) &+ \operatorname{res}\big(f^*(s)t^{-s}; s = 0\big) \\ &- \frac{1}{2\pi i}\int_{c+i\infty}^{-1/2+i\infty} f^*(s)t^{-s}\,ds \\ &- \frac{1}{2\pi i}\int_{-1/2+i\infty}^{-1/2-i\infty} f^*(s)t^{-s}\,ds \\ &- \frac{1}{2\pi i}\int_{-1/2-i\infty}^{c-i\infty} f^*(s)t^{-s}\,ds. \end{aligned}$$

The required residues are easily evaluated. At $s = 1$, we have a simple pole from $\zeta(s)$. Using the well-known expansions

$$\begin{aligned} \zeta(s) &= \frac{1}{s-1} + \gamma + O(s-1), \\ \zeta(s+1) &= \frac{\pi^2}{6} + O(s-1) \end{aligned}$$

for $s \to 1$, we find

$$f^*(s)t^{-s} = -\frac{\pi^2}{6t}\frac{1}{s-1} + O(1) \qquad \text{for } s \to 1.$$

At $s = 0$, we obtain a double pole resulting from $\Gamma(s)$ and $\zeta(s+1)$. Since

$$\begin{aligned} \Gamma(s) &= \frac{1}{s} - \gamma + O(s), \\ \zeta(s) &= -1/2 - \frac{\log 2\pi}{2}s + O(s^2), \\ t^{-s} &= e^{-s\log t} = 1 - s\log t + O(s^2) \end{aligned}$$

for $s \to 0$, we easily obtain the expansion

$$f^*(s)t^{-s} = \frac{1}{2s^2} - \Big(\frac{\log t}{2} - \frac{\log 2\pi}{2} + \frac{\gamma}{2} - \frac{\gamma}{2}\Big)\frac{1}{s} + O(1) \qquad \text{for } s \to 0.$$

Applying certain standard estimations, it is possible to show that the integrals along the horizontal segments provide vanishing contributions towards $\pm i\infty$, and that the integral along the vertical contour $\Re(s) = -1/2$ contributes a remainder term $O(t^{1/2})$ to the asymptotic expansion of $f(t)$; see [**VF**] for details. Therefore we obtain

$$f(t) = -\frac{\pi^2}{6t} - \frac{\log t}{2} + \frac{\log 2\pi}{2} + O(t^{1/2}) \qquad \text{for } t \to 0+.$$

Note that we may improve the remainder term to $O(t)$, since the next singularity (a simple pole) of the transform lies at $s = -1$. Moreover, it would be easy to derive a full asymptotic expansion by shifting the horizontal contour further left.

Finally, since $t = -\log(1-p) = p + O(p^2)$ and $\log t = \log(-\log(1-p)) = \log p + O(p)$ for $p \to 0$, equation (A.1) establishes the asymptotic expansion as asserted in Lemma 4.1. ∎

Proof of Lemma 5.2: Bearing in mind the connection between F and G_Λ mentioned after the introduction of system (4.1), we obtain the upper bound

$$\begin{aligned}
\big(G_\Lambda(\mathbf{a})\big)^{(n)} - \big(F(\mathbf{a})\big)^{(n)} &= e^{-\lambda} \sum_{k\geq 2} \frac{\lambda^k}{k!}(a_{n+1} - a_{n+k}) \\
&= e^{-\lambda} \sum_{k\geq 2} \frac{\lambda^k}{k!} \sum_{j=1}^{k-1} (a_{n+j} - a_{n+j+1}) \\
&= e^{-\lambda} \sum_{j\geq 1} (a_{n+j} - a_{n+j+1}) \sum_{k\geq j+1} \frac{\lambda^k}{k!} \\
&= e^{-\lambda}(a_{n+1} - a_{n+2}) \sum_{j\geq 1} \prod_{i=n+2}^{n+j} \bar{q}_i \sum_{k\geq j+1} \frac{\lambda^k}{k!} \\
&\leq e^{-\lambda}(a_{n+1} - a_{n+2}) \sum_{j\geq 1} \lambda^{-(j-1)} \frac{\lambda^{j+1}}{(j+1)!} \sum_{k\geq 0} \frac{(j+1)!}{(k+j+1)!} \lambda^k \\
&< e^{-\lambda}(a_{n+1} - a_{n+2}) \frac{\lambda^2}{2} \sum_{j\geq 1} \frac{1}{(j-1)!} e^{\lambda} \\
&= \frac{e}{2} \lambda^2 (a_{n+1} - a_{n+2}).
\end{aligned}$$

The lower bound follows trivially from the first line equation. ∎

Proof of Lemma 5.3: Introducing the notation

$$g_n(\mu) = e^{-\mu}\Big(np(1-p)^{n-1}a_{n-1} + \big(\mu(1-p)^n + 1 - np(1-p)^{n-1}\big)a_n + \big(e^\mu - 1 - \mu(1-p)^n\big)a_{n+1}\Big),$$

we have $g_n(\phi_n) = \big(G_\Phi(\mathbf{a})\big)^{(n)}$ and $g_n(\lambda) = \big(G_\Lambda(\mathbf{a})\big)^{(n)}$. Therefore,

$$\big(G_\Lambda(\mathbf{a})\big)^{(n)} - \big(G_\Phi(\mathbf{a})\big)^{(n)} \geq g_n'(\phi_n)(\lambda - \phi_n) + \min_{\lambda \leq \mu \leq \phi_n} g_n''(\mu) \frac{(\lambda - \phi_n)^2}{2}.$$

Now, providing

$$\begin{aligned}
g_n'(\mu) &= -e^{-\mu}\Big(np(1-p)^{n-1}(a_{n-1} - a_n) + \big(1 - (1-p)^n + \mu(1-p)^n\big)(a_n - a_{n+1})\Big) \\
g_n''(\mu) &= e^{-\mu}\Big(np(1-p)^{n-1}(a_{n-1} - a_n) + \big(1 - 2(1-p)^n + \mu(1-p)^n\big)(a_n - a_{n+1})\Big)
\end{aligned}$$

yields

$$g_n'(\phi_n) = -e^{-\phi_n}(a_n - a_{n+1})\Big(np(1-p)^{n-1}\frac{1}{\bar{q}_n} + \big(1-(1-p)^n + \phi_n(1-p)^n\big)\Big)$$
$$= -\big(1 - e^{-\phi_n}(1-p)^n\big)(a_n - a_{n+1})$$

and, by almost the same computation,

$$g_n''(\mu) = e^{-\mu}(a_n - a_{n+1})(e^{\phi_n} + \mu(1-p)^n - (2+\phi_n)(1-p)^n)$$
$$\geq -e^{-\mu}(a_n - a_{n+1})$$

since

$$e^{\phi_n} + \mu(1-p)^n - (2+\phi_n)(1-p)^n \geq 1 + \phi_n - 2 - \phi_n = -1.$$

We therefore obtain

$$-g_n'(\phi_n)(\phi_n-\lambda) + \min_{\lambda\leq\mu\leq\phi_n} g_n''(\mu)\frac{(\phi_n-\lambda)^2}{2} \geq (a_n - a_{n+1})(\phi_n-\lambda)\Big(1 - e^{-\phi_n}(1-p)^n - \frac{\phi_n-\lambda}{2}\Big)$$

and an additional, trivial minorization completes the proof of Lemma 5.3. ∎

Proof of Lemma 6.3: First notice that

$$b_{n-1} \geq e^{-\lambda}\Big(np(1-p)^{n-1}b_{n-2} + \big(1 - np(1-p)^{n-1} + \lambda(1-p)^n\big)b_{n-1}$$
$$+ \big(e^{\lambda} - 1 - \lambda(1-p)^n\big)b_n\Big)$$

is equivalent to $q_n \leq q_{n-1}$, cf. definition (4.3), and that the latter is easily established for $n \geq 1/p$. So we trivially get

$$b_{n-1} - \big(\mathrm{L}(b_{n-1})_{n\geq 1}\big)^{(n)} \geq e^{-\lambda}\frac{\lambda^2}{2}(b_n - b_{n+1}),$$

recall the proof of the lower bound of Lemma 5.2. Since $b_n - b_{n+1} = (1-b_1)P_n$ according to our derivations in Section 4 and therefore

$$Q_n \leq b_n = (1-b_1)P_n(1 + q_{n+1} + q_{n+1}q_{n+2} + \cdots),$$

we only have to show that there is a constant $C > 0$ such that

$$\frac{2e^{\lambda}}{\lambda^2}(1 + q_{n+1} + q_{n+1}q_{n+2} + \cdots) \leq \exp\Big(\frac{C}{p}\log\lambda^{-1}\Big)$$

for $n \geq n_0$. But this is an easy exercise: For $j > n_0$ we have

$$q_j \leq \frac{n_0 p\lambda}{e^{\lambda} - 1 - \lambda^2} = O(\log\lambda^{-1})$$

and for $j \geq n_0 + \frac{2}{p} \log\log \lambda^{-1}$

$$q_j = O\Big(\log \lambda^{-1} (1-p)^{\frac{2}{p}\log\log\lambda^{-1}}\Big) = O\Big(\frac{1}{\log \lambda^{-1}}\Big).$$

Hence we obtain

$$\begin{aligned} 1 + q_{n+1} + q_{n+1}q_{n+2} + \cdots &\leq \frac{2}{p} \log\log \lambda^{-1} \left(C_1 \log \lambda^{-1}\right)^{\frac{2}{p}\log\log\lambda^{-1}} \Big(1 + O(1)\Big) \\ &\leq \exp\Big(\frac{C_2}{p} (\log\log \lambda^{-1})^2\Big) \end{aligned}$$

for $n \geq n_0$, and $2e^{\lambda}/\lambda^2 \leq \exp\big(C_3 \log \lambda^{-1}\big)$ finally establishes Lemma 6.3. ∎

Proof of Lemma 6.5: First, certain straightforward algebraic manipulations easily establish that $c_n - (\mathrm{Lc})^{(n)} \geq 1$ is equivalent to

$$\begin{aligned} c_n - c_{n-1} \geq \frac{1}{np(1-p)^{n-1}} \Big(e^{\lambda} &+ \big(e^{\lambda} - 1 - \lambda(1-p)^n\big)(c_{n+1} - c_n) \\ &+ \sum_{j \geq 2} \frac{\lambda^j}{j!} (c_{n+j} - c_{n+1})\Big). \end{aligned} \tag{A.2}$$

We start proving (A.2) by an appropriate upper bound for the last term on the right hand side. Mentioning that $a_{k-1} \leq a_k$ for $1 \leq k \leq n_1$, a similar derivation as in the proof of Lemma 5.2 yields

$$\begin{aligned} \sum_{j\geq 2} \frac{\lambda^j}{j!}(c_{n+j} - c_{n+1}) &= \sum_{k \geq 1} (c_{n+k+1} - c_{n+k}) \sum_{j > k} \frac{\lambda^j}{j!} \\ &\leq \sum_{k=1}^{n_1-n-1} a_{n_1-n-k} \frac{\lambda^{k+1}}{(k+1)!} e^{\lambda} + \sum_{k=n_1-n}^{\infty} a_0 2^{n+k-n_1} \frac{\lambda^{k+1}}{(k+1)!} e^{\lambda} \\ &\leq \sum_{k=1}^{n_1-n-1} a_{n_1-n} \Big(\frac{C_4 \lambda^2}{p}\Big)^{-k} n_1^k \frac{\lambda^{k+1}}{(k+1)!} e^{\lambda} + a_0 \frac{\lambda^{n_1-n+1}}{(n_1-n+1)!} e^{3\lambda} \\ &= O(a_{n_1-n}\lambda^2) \end{aligned}$$

for $n < n_1$, uniformly for $C_4 \geq C_2$. Furthermore, we easily obtain

$$\begin{aligned} (1-p)^n &\geq 1 - C_5 \lambda^2, \\ e^{\lambda} - 1 - \lambda(1-p)^n &= O(\lambda^2), \\ e^{\lambda} &\leq a_0 \big(e^{\lambda} - 1 - \lambda(1-p)^n\big) \leq a_{n_1-n}\big(e^{\lambda} - 1 - \lambda(1-p)^n\big) \end{aligned}$$

for $n < n_1$, if $C_3 > 2$ is chosen properly.

Putting everything together it follows that the right hand side of inequality (A.2) is bounded above by

$$\frac{C_4\lambda^2}{np}(c_{n+1} - c_n)$$

for some $C_4 > C_2$. Finally, mentioning that our c_n from (6.6) exactly solve

$$c_n - c_{n-1} = \frac{C_4\lambda^2}{np}(c_{n+1} - c_n) \qquad \text{for } n < n_1$$

with $c_0 = 0$ and $c_{n_1} - c_{n_1-1} = a_0$, the proof of Lemma 6.5 is complete. ∎

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