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Static Priority Scheduling of Aperiodic Real-Time Tasks

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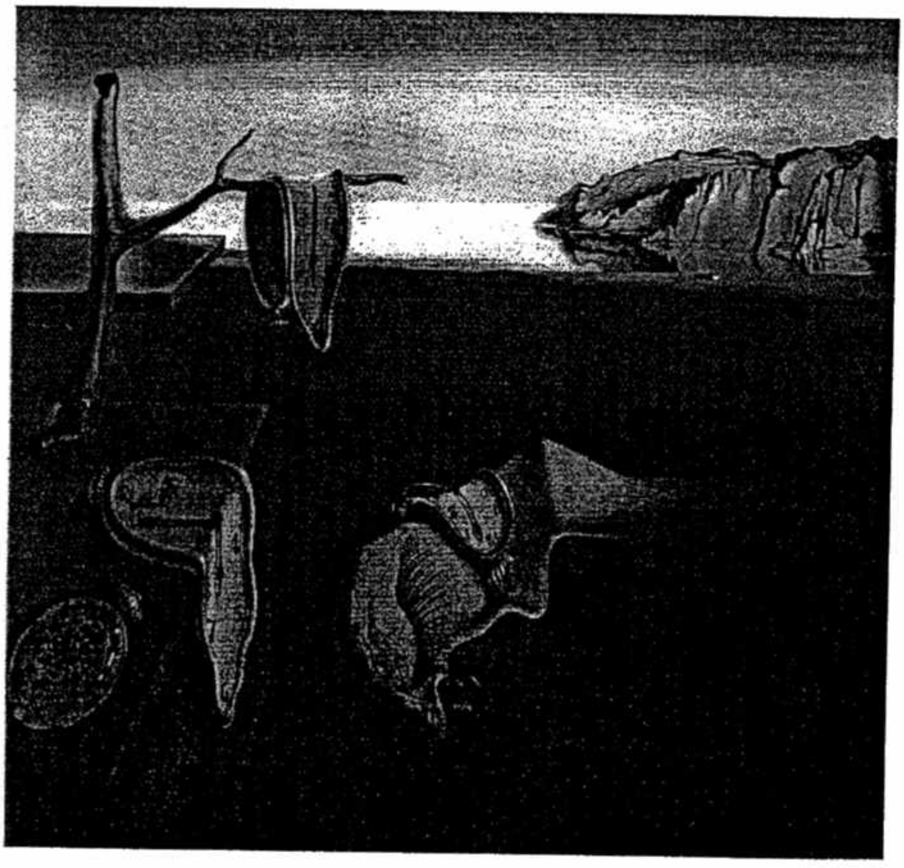


Salvador Dali, "Die Beständigkeit der Erinnerung"

STATIC PRIORITY SCHEDULING OF APERIODIC REAL-TIME TASKS

ULRICH SCHMID

ABSTRACT. We investigate deadline meeting properties of the well-known (preemptive) static priority scheduling (SPS) algorithm, which is widespreadly used in commercial real-time operating system kernels. A discrete-time single server queueing system employing SPS for scheduling probabilistically arriving tasks at L priority levels is considered for this purpose. Model parameters are arrival and execution-time distribution $A_\ell(z)$, $L_\ell(z)$ and a (constant) deadline $T_\ell \in \mathcal{T}_L$ per level ℓ. By means of a combinatorial technique (which does not require stable-state assumptions), we determine the probability distribution of the (random-)time the system operates without violating any task's deadline. This distribution is asymptotically exponential with parameter $\lambda_{\mathcal{T}_L}$, which decreases exponentially with the deadlines $\mathcal{T}_L$; simple asymptotic expressions for $\lambda_{\mathcal{T}_L}$ and all associated quantities (probabilities, moments, ...) for large $\mathcal{T}_L$ are provided. Our numerical examples suggest that real-time systems based on SPS operate reasonably well only if computing performance is (more than) adequate.

1. INTRODUCTION

Real-time computer systems as found in spacecrafts, power plants, or automated factories are playing a more and more vital role in our daily life. Despite of the increasing criticalness of operation, however, they are somewhat neglected by traditional computer science, in particular, by classical performance evaluation research.

Generally speaking, tasks of a real-time system have to be performed not only in a correct, but also in a timely fashion; usually, they must finish within a predefined deadline. Otherwise, there might be more (*hard real-time*) or less (*soft real-time*) severe consequences. One of the most important problems in the design of real-time systems concerns methods for a suitable *task scheduling*, see [TK91], [CSR88] for a survey. Scheduling goals for real-time systems, however, are different from those fitting the needs of ordinary computer systems, since timeliness is not a simple consequence of high throughput or similar performance characteristics.

Our research in that area aims at investigating deadline meeting properties of scheduling algorithms for probabilistically arriving (*"aperiodic"*) tasks in soft real-time systems, see [S95] for a comprehensive overview. In our earlier work, we identified a characteristic quantity (we called it *successful run duration*) that allows comparison of the real-time performance of different scheduling algorithms. In this paper, we investigate the successful run duration of the important *preemptive static priority scheduling algorithm* (SPS), a scheme that is widely used in practical applications due to its inherent simplicity. Actually, almost all existing real-time operating system kernels like VRTX (Ready Systems) or pSOS$^+$ (Integrated Systems/SCG), for example, are built upon SPS. Our research is —to the best of our knowledge— the first attempt to quantify how well they are actually performing.

Key words and phrases. Real-time behaviour, static priority scheduling, deadlines, aperiodic tasks, combinatorial probabilities, trees, bivariate singularity analysis, asymptotics.

The rest of this paper is organized as follows: Section 2 briefly introduces the underlying queueing system model and the parameters of interst. Section 3 is devoted to the combinatorial aspects of our investigations, Section 4 contains the asymptotic analysis. A discussion of our results, including a numerical example and some directions of further research, in Section 5 eventually complete the paper.

2. System Model

Our system model is based on a queueing system consisting of a task scheduler, a task list of (potential) infinite capacity, and a single server. Newly arriving tasks are inserted into the task list by the static priority scheduling algorithm, which works as follows: The task list of our system is sorted according to descending priorities. We assume that there are $L \geq 1$ different *priority levels* numbered from $1, \ldots, L$, where 1 is the highest priority one. A newly arriving task of a certain priority level is inserted into the task list behind the already queued tasks of the same level. If the task list becomes empty, a (short) dummy task is generated by the scheduler. The server always executes the task at the head of the task list, in a preemptible fashion. If the server processes a dummy task, the system is called *idle*, otherwise *busy*.

Rearranging of the task list (preemption/scheduling) occurs at discrete points in time only, without any overhead. The length of the interval between two such points is an integral multiple of some unit time called a (machine) *cycle*. Due to this assumption, we are able to model tasks formed by non-preemptible *actions* with duration of 1 cycle. The *task execution time* of a task is the number of cycles necessary for processing the task to completion if the server is exclusively available. A dummy task consists of a single no-operation action (1 cycle), an ordinary task may have an arbitrary task execution time. More specifically, the *probability generating function* (PGF) of level-ℓ task execution times (measured in cycles) is denoted by

$$L_\ell(z) = \sum_{k\geq 1} l_{k,\ell} z^k, \text{ where } l_{k,\ell} = \mathbf{P}\{\text{task execution time of level-}\ell\text{ task is } k \text{ cycles}\}, \quad (2.1)$$

with the additional assumption $L_\ell(0) = 0$, i.e., all task execution times must be greater than or equal to one cycle. Note that the above definition implies that task execution times are independent of each other and independent of task arrivals.

The PGF of the number of level-ℓ task arrivals during a cycle is denoted by

$$A_\ell(z) = \sum_{k\geq 0} a_{k,\ell} z^k, \text{ where } a_{k,\ell} = \mathbf{P}\{k \text{ tasks of level } \ell \text{ arrive during a cycle}\}, \quad (2.2)$$

and it should meet the constraint $a_{0,\ell} = A_\ell(0) > 0$, i.e., the probability of no arrivals during a cycle should be greater than zero. This in fact ensures the existence of idle cycles. The above definition implies the independence of arrivals in two arbitrary different cycles and at different priority levels.

Since it turns out that the overall execution time, i.e., the number of cycles necessary for processing all actions induced by task-ℓ arrivals within a cycle, plays a central role, we introduce

$$P_\ell(z) = \sum_{k\geq 0} p_{k,\ell} z^k = A_\ell\big(L_\ell(z)\big), \quad (2.3)$$

with the following additional assumptions:

(1) The average number of actions induced by task-ℓ arrivals within a cycle should be smaller than one, i.e., $0 < P_\ell'(1) < 1$. Note that this implies $p_{0,\ell} = P_\ell(0) > 0$ since $1 > P_\ell'(1) \geq P_\ell(1) - p_{0,\ell} = 1 - p_{0,\ell}$.
(2) $P_\ell''(z) \not\equiv 0$, i.e., we explicitly exclude the trivial case $P_\ell(z) = p_{0,\ell} + (1 - p_{0,\ell})z$.
(3) The radius of convergence R_{P_ℓ} of $P_\ell(z)$ should be larger than 1, which implies that all moments are finite. Moreover, we require that $P_\ell(x)$ gets sufficiently large for x large enough; a sufficient condition providing this is $\lim_{x \to R_{P_\ell}-} P_\ell(x) = +\infty$.

In addition to the conditions above, we also need a "global" constraint: The load of the system must be less than 100% on the average, that is, we assume that our system has to deal with task arrivals keeping it not totally busy on the average. This may be expressed by

$$\sum_{\ell=1}^{L} P_\ell'(1) < 1, \tag{2.4}$$

since $P_\ell'(1)$ equals the average number of actions caused by level-ℓ task arrivals within a cycle.

We should mention here that the number of *globally* valid probability distributions meeting our constraints is of course considerably limited due to the required independency. Globally valid distributions consistent with our assumptions must be based on an interarrival distribution with the memoryless property, i.e., an exponential or geometric distribution, leading to (well-thumbed) Poisson- or Bernoulli-type arrivals, respectively.

The overall operation of our system may be viewed as a sequence of *busy periods*, consisting of a single initial idle cycle and zero or more busy cycles each. We call a busy period *feasible*, if all tasks processed during the busy period meet their service time deadline. Herein, the *service time* of a task is the time (measured in cycles) from the beginning of the cycle in which the task arrives at the system to the end of the cycle which completes the execution of the task. For each priority level ℓ, we assume a constant *service time deadline* $T_\ell \geq 2$. Finally, a sequence of feasible busy periods followed by a non-feasible busy period (containing at least one deadline violation) is called a *run*, the sequence without the last (violating) busy period is refered to by a *successful run*.

The random variable *successful run duration*, which is the time interval from the beginning of an initial idle cycle to the beginning of the (idle) cycle starting the busy period containing the very first violation of a task's deadline, was found to be a suitable —in particular, mathematically tractable— measure for assessing the real-time performance of a scheduling algorithm, see [S95], [DS93], [BS92], [SB92], [BS91], [SB94]. Note that, unlike conventional queueing theory, this approach does not need stable-state assumptions. In [DS93] we showed[1] that, under a few conditions (which follow easily from the ones mentioned above), the successful run duration is approximately (asymptotically) exponentially distributed with a parameter equal to the reciprocal of the average length of a single feasible busy period. Therefore, the analysis of the distribution of the successful run duration for an arbitrary scheduling algorithm boils down to a relatively straightforward average case analysis.

Introducing the abbreviations $\mathcal{T}_L = \{T_L, \ldots, T_1\}$ and $\mathcal{T}_{L-1} = \{T_{L-1}, \ldots, T_1\}$, the problem

[1] By using a very similar approach, we also solved the old problem of analyzing the duration of the successful operation of the well-known slotted ALOHA collision resolution algorithm, which we found exponentially distributed too; see [DS93b] and [Drm91] for details.

of investigating SPS reduces to the derivation of

$$\mu_{T_L}^{[L]} = \frac{B_{T_L}^{[L]\prime}(1)}{1 - B_{T_L}^{[L]}(1)}, \tag{2.5}$$

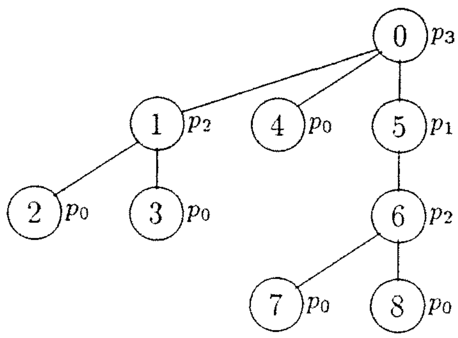
where $B_{T_L}^{[L]}(z)$ is the improper[2] PGF of the length of a single *T_L-feasible busy period*. The derivation of $B_{T_L}^{[L]}(z)$ relies on combinatorial techniques applied to certain random trees representing feasible busy periods. Convenient asymptotic expressions for $\mu_{T_L}^{[L]}$ are eventually obtained by singularity analysis of the PGFs involved.

3. Combinatorics

Our starting point for the combinatorial analysis is the one-to-one correspondence between (feasible) busy periods and a family of width-constrained trees introduced in the investigation of *first come first served* (FCFS) scheduling without priorities, cf. [SB92], which we briefly review below. Note that static priority scheduling implies that tasks of the same priority level are queued in FCFS order, whereas higher priority ones are queued in (preemptive) *last come first served* (LCFS) order, which allows us to use some of the ideas of the analysis of FCFS scheduling.

Dealing with FCFS allows us to consider an *expanded tasks list*, which contains all the actions the tasks in the original task list consist of. After all, for the purpose of analysis, we can be clairvoyant w.r.t. task execution times and even further arrivals. Anyway, it is easy to see that if the length of the expanded task list is always bounded by $T-1$ during a busy period, then it is guaranteed that the busy period is T-feasible (and vice versa). Note that shifting by 1 is a consequence of our definition of service time, which starts at the beginning of the initial cycle.

Consider the example arrival sequence $3, 2, 0, 0, 0, 1, 2, 0, 0, \ldots$, which gives the number d_n of actions to be executed due to task arrivals during the initial, first, second ... cycle of a busy period; e.g., in the initial (idle-)cycle, there are task arrivals with a total task execution time of $d_0 = 3$ cycles. The construction of the corresponding tree works as follows: Each *node* represents a cycle n of the busy period; the root corresponds to the initial cycle 0. A node n has d_n successors, according to the number of actions (i.e., cycles) caused by arrivals during the corresponding cycle n, and is weighted by the appropriate probability p_{d_n}, cf. (2.3). For some reason which will become clear soon, we use the following "aligned" representation of the tree corresponding to the arrival sequence above:



[2]That is, $B_{T_L}^{[L]}(1) < 1$ since we restrict ourselves to feasible busy periods.

Each node is also labeled by the number of the corresponding cycle within the busy period. This labeling is obtained by a preorder traversal (left to right) of the tree. Consider now e.g. the node with label 1: At the beginning of the corresponding cycle 1 in the busy period, the initial (idle) action has just left the task list and the first action of the task(s) that arrived during the initial cycle is to be executed. One encounters that, due to our special alignment, the horizontal distance of that node from the right margin is equal to the length of the task list at the time the corresponding cycle of the busy period is executed. A short reflection should make it clear that this statement is true for all nodes in the tree. Therefore, limiting the service times by a deadline T corresponds to limiting the width of the aligned tree to $T-1$ vertices.

We omit restating the symbolic equation describing this family of trees and its translation to the ordinary generating function (OGF) from [SB92], since FCFS is a special case ($L=1$) of SPS. However, to introduce the general technique (see [VF90] for a nice overview), we establish some results on general busy periods required in §5. More specifically, we consider the (of course well-known, see e.g. [Fel68, p. 298], [BS92]) situation where no deadline restrictions are present. In this case, the duration of a busy period does not depend on the scheduling discipline but only on the task arrivals. We have the following symbolic equation for the family $\mathcal{B}^{[L]}$ of corresponding trees:

$$\mathcal{B}^{[L]} = p_0^{[L]} \bigcirc \; + \; p_1^{[L]} \begin{array}{c}\bigcirc\\ \mathcal{B}^{[L]}\end{array} + \cdots + \; p_k^{[L]} \begin{array}{c}\bigcirc\\ \underbrace{\mathcal{B}^{[L]} \ldots \mathcal{B}^{[L]}}_{k}\end{array} + \cdots \qquad (3.1)$$

Note that those trees do not preserve the scheduled execution order.

The probability weights $p_k^{[L]}$ denote the probability that the total number of actions arising from tasks arriving during the initial cycle (denoted by $\bigcirc$) equals k. Since arrivals and task execution times are independent, we obviously have

$$p_k^{[L]} = [z^k]P^{[L]}(z) \qquad \text{for } k \geq 0,$$

where

$$P^{[L]}(z) = \prod_{\ell=1}^{L} P_\ell(z). \qquad (3.2)$$

We are interested in the probability $b_n^{[L]}$ that a random tree in $\mathcal{B}_L^{[L]}$ has size n, with the size being the number of nodes in the tree. More specifically, we are looking for the PGF

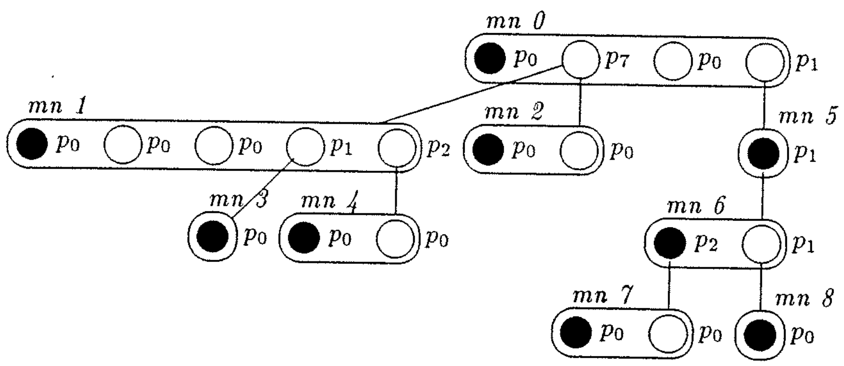
$$B^{[L]}(z) = \sum_{n\geq 0} b_n^{[L]} z^n. \qquad (3.3)$$

Since probability weights have the same compositional properties as counting weights (the probability of the union and intersection of two disjoint and independent events equals the sum and the product, respectively, as it is the case for cardinalities of sets), the whole theory of translating admissible combinatorial constructions to the corresponding ordinary generating functions (which are obviously PGFs in this case) apply. Accordingly, we have to mark each node with the counting variable z and to apply straightforward product and sum translations to obtain the well-known functional equation (cf. [Fel68, p. 298], [MM78], [BS92])

$$B^{[L]}(z) = z \sum_{k \geq 0} p_k^{[L]} \Big(B^{[L]}(z)\Big)^k = zP^{[L]}\Big(B^{[L]}(z)\Big). \tag{3.4}$$

We will summarize these (and other) results in Theorem 3.2 at the end of this section.

Now we will start with the extensions of the FCFS tree model required for multiple priority levels. The basic idea is simple: Since tasks of the same priority level are queued in FCFS order, whereas higher priority ones are handled according to the (preemptive) LCFS strategy, we just have to "blow" up nodes in such a way that each of the resulting *multi-nodes* corresponds to a single higher priority busy period. For example, assuming that the earlier FCFS-tree represents arrivals at priority level 2, we let each node of that tree correspond to the whole busy period initiated by those level-1 tasks that arrive during the original level-2 cycle (which is thus the initial cycle of this level-1 busy period). The following tree should illustrate this procedure:



Each multi-node consists of the (black) initial node that belongs to a level-2 task, and zero or more ordinary nodes making up the busy period of higher priority tasks arriving during (and after) the initial cycle. As far as level-2 deadlines are concerned, it is solely the overall number of nodes in a multi-node that counts. We thus arranged all nodes in a multi-node in the sequence of their execution from left to right (with the initial node being the left-most one). It is understood, however, that each multi-node actually corresponds to an aligned tree involving level 1 tasks (and, in case of even higher priority levels, higher level ones "hidden" in their multi-nodes).

Global execution order is constructed from the tree by a (left to right) preorder traversal of all circular nodes, as in FCFS. This means, for example, that the task arrival in the 4th (rightmost) node in the root multi-node $mn0$ does not take place during the 4th cycle of the root busy period, but rather during the last cycle of the busy period represented by $mn2$. Therefore, our tree does not reflect the global execution order, but rather something like a local arrival order w.r.t. each individual cycle. Actually, multi-node boundaries are meaningful for the immediate successors of a single cycle only. All the actions corresponding to their nodes are appended *at the end* of the task list present at the time when the predecessor cycle is executed. From then on, they are just anonymous actions that will be executed before newly arriving ones. Note, however, that this makes sense only when task arrivals in different cycles are independent of each other, recall (2.2).

The fact that our tree does not reflect the global execution order causes another intricacy connected with initial (black) nodes of a multi-node. To correctly model the fact that level-2 arrivals during the initial cycle must wait until a possibly simultaneously initiated higher priority busy period has terminated, we had to arrange initial nodes as leftmost ones. However, they belong to a level-2 task and are obviously executed *before* their higher-priority neighbors. Hence, all outer leftmost multi-nodes in our tree require special attention: They may exceed the level-2 width-constraint provided that there are no level-2 arrivals in those nodes of the multi-node that lie beyond the limit.

Our aligned trees thus guarantee that the distance of a node from the right margin represents the length of the task list at the time the corresponding cycle is executed. Consequently, limiting e.g. level-2 service times by a deadline T_2 corresponds to limiting the width of the aligned tree to $T_2 - 1$ vertices. For technical reasons, however, it is more convenient to deal with the family of aligned trees that are width-constrained by T_ℓ instead of $T_\ell - 1$, shifting back everything by 1 at the end. Recalling the abbreviation $\mathcal{T}_L = \{T_L, \ldots, T_1\}$, we consider the family $\mathcal{C}^{[L]}_{\mathcal{T}_L}$ of $\mathcal{T}_L$-width trees with the OGF (which is an improper PGF)

$$C^{[L]}_{\mathcal{T}_L}(z) = \sum_{n \geq 0} c^{[L]}_{n,\mathcal{T}_L} z^n \tag{3.5}$$

where

$$c^{[L]}_{n,\mathcal{T}_L} = \mathbf{P}\{\text{total number of nodes of a random } \mathcal{T}_L\text{-width tree is } n\}.$$

Note that $c^{[L]}_{0,\mathcal{T}_L} = 0$ since there is always a root node in any tree. For notational convenience, we use abbreviations like the above ones where possible.

Our aim is to provide an equation for $C^{[L]}_{\mathcal{T}_L}(z)$ that involves $C^{[L-1]}_{\mathcal{T}_{L-1}}(z)$, i.e., a recursive formula. However, we first derive a symbolic equation of similar width-constrained trees $\overline{\mathcal{C}}^{[L]}_{\mathcal{T}_L}$ with L priority levels, which are solely generated by level-L arrivals during the initial cycle; the connection to the actually desired family $\mathcal{C}^{[L]}_{\mathcal{T}_L}$ will be established subsequently. For further notational convenience, we introduce the additional abbreviation $\mathcal{C}^{[L]}_k = \mathcal{C}^{[L]}_{k,T_{L-1},\ldots,T_1}$; note that $\mathcal{C}^{[L]}_{T_L} = \mathcal{C}^{[L]}_{\mathcal{T}_L}$.

$$\begin{aligned}
\overline{\mathcal{C}}^{[L]}_{\mathcal{T}_L} = {} & \\
& \overline{p}^{[L]}_0 \bigcirc + \overline{p}^{[L]}_1 \begin{array}{c}\bigcirc\\ \overline{\mathcal{C}}^{[L]}_{T_L}\end{array} + \cdots + \overline{p}^{[L]}_k \begin{array}{c}\bigcirc\\ \overline{\mathcal{C}}^{[L]}_{T_L-k+1}\cdots\overline{\mathcal{C}}^{[L]}_{T_L-1}\overline{\mathcal{C}}^{[L]}_{T_L}\end{array} + \cdots + \overline{p}^{[L]}_{T_L} \begin{array}{c}\bigcirc\\ \overline{\mathcal{C}}^{[L]}_{1}\cdots\overline{\mathcal{C}}^{[L]}_{T_L-1}\overline{\mathcal{C}}^{[L]}_{T_L}\end{array} \\
& + \overline{v}^{[L]}_{T_L+1,T_L} \begin{array}{c}\bigcirc\\ \mathcal{E}\overline{\mathcal{C}}^{[L]}_{1}\overline{\mathcal{C}}^{[L]}_{2}\cdots\overline{\mathcal{C}}^{[L]}_{T_L}\end{array} + \cdots + \overline{v}^{[L]}_{T_L+m,T_L} \begin{array}{c}\bigcirc\\ \underbrace{\mathcal{E}\cdots\mathcal{E}}_{m}\overline{\mathcal{C}}^{[L]}_{1}\overline{\mathcal{C}}^{[L]}_{2}\cdots\overline{\mathcal{C}}^{[L]}_{T_L}\end{array} + \cdots
\end{aligned} \tag{3.6}$$

In the equation above, $\mathcal{E}$ denotes a single cycle with no level-L arrivals; its OGF is clearly $E(z) = p_{0,L} z$.

It is easy to provide the required probability weights $\overline{p}^{[L]}_k$, which denote the probability that exactly k new actions arise as a consequence of (1) $m \geq 0$ level-L arrivals (but no higher priority ones) during the initial cycle and (2) all higher-priority arrivals during the m arising (level-L-)successors. With $C_{T_{L-1}}^{[L-1]}(z)$ denoting the OGF of width-constrained trees for higher priority levels $L-1,\ldots,1$, it is clear that

$$\overline{p}_k^{[L]} = [z^k]P_L\Big(C_{T_{L-1}}^{[L-1]}(z)\Big) \qquad \text{for } k \geq 0; \tag{3.7}$$

note that $\overline{p}_0^{[L]} = p_{0,L}$.

However, it is more complicated to provide the probability weights $\overline{v}_{k,T_L}^{[L]}$, $k > T_L \geq 1$, which reflect the situation of outer leftmost multi-nodes exceeding a width of T_L, as mentioned earlier. For our argument, we use two counting variables y, w to take care of all cycles (y) and those that are meaningful for level-L deadlines only (w). Now, while all inner multi-nodes that arrived during the initial cycle contribute to both y and w, we have to replace the outer leftmost multi-node $C_{T_{L-1}}^{[L-1]}(yw)$ by a special multi-node, say, $C(w,y)$, that contributes differently to w and y. More specifically, we require

$$C(w,y) = \sum_{n\geq 1} w^n \sum_{k\geq n} c_{k,T_{L-1}}^{[L-1]} y^k = \sum_{k\geq 1} c_{k,T_{L-1}}^{[L-1]} y^k \sum_{n=1}^{k} w^n = \frac{w\Big(C_{T_{L-1}}^{[L-1]}(y) - C_{T_{L-1}}^{[L-1]}(yw)\Big)}{1-w}$$

so that $[y^k][w^n]C(w,y) = c_{k,T_{L-1}}^{[L-1]}$ if $k \geq n \geq 1$ (and zero otherwise). But now it is clear that for $k > n \geq 1$

$$\begin{aligned}
\overline{v}_{k,n}^{[L]} &= \mathbf{P}\{k \text{ actions arise with } n \text{ being the last level-}L \text{ "relevant" one}\} \\
&= [y^k][w^n]\frac{P_L\Big(C_{T_{L-1}}^{[L-1]}(yw)\Big) - p_{0,L}}{C_{T_{L-1}}^{[L-1]}(yw)} \cdot \frac{w}{1-w}\Big(C_{T_{L-1}}^{[L-1]}(y) - C_{T_{L-1}}^{[L-1]}(yw)\Big) \\
&= [y^k][w^n]\frac{P_L\Big(C_{T_{L-1}}^{[L-1]}(yw)\Big) - p_{0,L}}{C_{T_{L-1}}^{[L-1]}(yw)} \cdot \frac{w}{1-w} \cdot C_{T_{L-1}}^{[L-1]}(y) \\
&= [y^k][w^n]\alpha(yw)\frac{w}{1-w}C_{T_{L-1}}^{[L-1]}(y);
\end{aligned} \tag{3.8}$$

$\alpha(yw)$ is used as an abbreviation. Note that we can indeed discard the term $C_{T_{L-1}}^{[L-1]}(yw)$ in the last but one step since, for any analytic function $f(.)$, we have $[y^k][w^n]\frac{w}{1-w}f(yw) = 0$ for $k \geq n$.

The translation of the symbolic equation for $\overline{C}_{T_L}^{[L]}$ into a functional equation of the OGFs involved yields

$$\overline{C}_{T_L}^{[L]}(z) = z\sum_{k=0}^{T_L} \overline{p}_k^{[L]} \prod_{i=T_L-k+1}^{T_L} \overline{C}_i^{[L]}(z) + z\prod_{i=1}^{T_L} \overline{C}_i^{[L]}(z) \sum_{m\geq 1} \overline{v}_{T_L+m,T_L}^{[L]}(zp_{0,L})^m. \tag{3.9}$$

Defining

$$\overline{Q}_n^{[L]}(z) = \frac{1}{\overline{C}_n^{[L]}(z)\cdots\overline{C}_1^{[L]}(z)} \qquad \text{for } n \geq 1,$$

$$\overline{Q}_0^{[L]}(z) \equiv 1,$$

and the corresponding bivariate generating function

$$\overline{Q}^{[L]}(s,z) = \sum_{n\geq 0} \overline{Q}_n^{[L]}(z)s^n, \tag{3.10}$$

multiplying (3.9) by $\overline{Q}^{[L]}_{T_L}(z)$ yields

$$\overline{Q}^{[L]}_{T_L-1}(z) = z\sum_{k=0}^{T_L} \overline{p}^{[L]}_k \overline{Q}^{[L]}_{T_L-k}(z) + z\sum_{m\geq 1} \overline{v}^{[L]}_{T_L+m,T_L}(zp_{0,L})^m.$$

This primarily involves a simple Cauchy product; multiplying both sides by s^{T_L} and summing up for $T_L \geq 1$, we find by using (3.7)

$$s\overline{Q}^{[L]}(s,z) = z\left(\overline{Q}^{[L]}(s,z)P_L\left(C^{[L-1]}_{T_{L-1}}(s)\right) - p_{0,L}\right) + zG(s,z)$$

and hence

$$\overline{Q}^{[L]}(s,z) = \frac{zp_{0,L} - zG(s,z)}{zP_L\left(C^{[L-1]}_{T_{L-1}}(s)\right) - s}.$$

By virtue of (3.8), we have

$$\begin{aligned}
G(s,z) &= \sum_{T_L\geq 1}\sum_{m\geq 1} \overline{v}^{[L]}_{T_L+m,T_L}(zp_{0,L})^m s^{T_L} \\
&= \sum_{T_L\geq 1}\sum_{m\geq 1} [y^{T_L+m}][w^{T_L}]\alpha(yw)\frac{w}{1-w}C^{[L-1]}_{T_{L-1}}(y)(zp_{0,L})^m s^{T_L} \\
&= \sum_{m\geq 1}[y^m]\left[\sum_{T_L\geq 1}[u^{T_L}]\alpha(u)\frac{u/y}{1-u/y}s^{T_L}\right]C^{[L-1]}_{T_{L-1}}(y)(zp_{0,L})^m \\
&= \sum_{m\geq 1}[y^m]\alpha(s)\frac{s/y}{1-s/y}C^{[L-1]}_{T_{L-1}}(y)(zp_{0,L})^m \\
&= \alpha(s)H^{[L]}(s,z),
\end{aligned}$$

where

$$\begin{aligned}
H^{[L]}(s,z) &= \sum_{m\geq 1}[y^m]\frac{s/y}{1-s/y}C^{[L-1]}_{T_{L-1}}(y)(zp_{0,L})^m \\
&= \sum_{m\geq 1}[y^m]C^{[L-1]}_{T_{L-1}}(y)\sum_{k\geq 1}\left(\frac{s}{y}\right)^k(zp_{0,L})^m \\
&= \sum_{m\geq 1}\sum_{k\geq 1}[y^{m+k}]C^{[L-1]}_{T_{L-1}}(y)s^k(zp_{0,L})^m \\
&= \sum_{n\geq 2}\sum_{k=1}^{n-1}[y^n]C^{[L-1]}_{T_{L-1}}(y)s^k(zp_{0,L})^{n-k} \\
&= \sum_{n\geq 2} c^{[L-1]}_{n,T_{L-1}}(zp_{0,L})^n\sum_{k=1}^{n-1}\left(\frac{s}{zp_{0,L}}\right)^k \qquad (3.11) \\
&= \sum_{n\geq 2} c^{[L-1]}_{n,T_{L-1}}(zp_{0,L})^n\left[\frac{1-\left(\frac{s}{zp_{0,L}}\right)^n}{1-\frac{s}{zp_{0,L}}} - 1\right] \\
&= \frac{1}{1-\frac{s}{zp_{0,L}}}\left[\frac{s}{zp_{0,L}}\left(C^{[L-1]}_{T_{L-1}}(zp_{0,L}) - c^{[L-1]}_{1,T_{L-1}}zp_{0,L}\right) - \left(C^{[L-1]}_{T_{L-1}}(s) - c^{[L-1]}_{1,T_{L-1}}s\right)\right]
\end{aligned}$$

$$= \frac{sC_{T_{L-1}}^{[L-1]}(zp_{0,L}) - zp_{0,L}C_{T_{L-1}}^{[L-1]}(s)}{zp_{0,L} - s}.$$

Remembering the abbreviation $\alpha(s)$ introduced in (3.8), we eventually obtain

$$\begin{aligned}
\overline{Q}^{[L]}(s,z) &= \frac{zp_{0,L}}{zP_L\left(C_{T_{L-1}}^{[L-1]}(s)\right) - s} - \frac{z\left[P_L\left(C_{T_{L-1}}^{[L-1]}(s)\right) - p_{0,L} + s/z - s/z\right]H^{[L]}(s,z)}{C_{T_{L-1}}^{[L-1]}(s)\left[zP_L\left(C_{T_{L-1}}^{[L-1]}(s)\right) - s\right]} \\
&= \frac{zp_{0,L}}{zP_L\left(C_{T_{L-1}}^{[L-1]}(s)\right) - s} - \frac{H^{[L]}(s,z)}{C_{T_{L-1}}^{[L-1]}(s)} + \frac{(zp_{0,L} - s)H^{[L]}(s,z)}{C_{T_{L-1}}^{[L-1]}(s)\left[zP_L\left(C_{T_{L-1}}^{[L-1]}(s)\right) - s\right]} \\
&= \frac{sC_{T_{L-1}}^{[L-1]}(zp_{0,L})}{C_{T_{L-1}}^{[L-1]}(s)\left[zP_L\left(C_{T_{L-1}}^{[L-1]}(s)\right) - s\right]} - \frac{H^{[L]}(s,z)}{C_{T_{L-1}}^{[L-1]}(s)} \\
&= \frac{Q^{[L]}(s,z) - H^{[L]}(s,z)}{C_{T_{L-1}}^{[L-1]}(s)},
\end{aligned} \tag{3.12}$$

where

$$Q^{[L]}(s,z) = \sum_{k\geq 0} Q_k^{[L]}(z)s^k = \frac{sC_{T_{L-1}}^{[L-1]}(zp_{0,L})}{zP_L\left(C_{T_{L-1}}^{[L-1]}(s)\right) - s}. \tag{3.13}$$

In order to obtain the desired width-constrained trees $\mathcal{C}_{T_L}^{[L]}$, we must take into account the possibility of higher-priority arrivals in the initial cycle (cf. our example tree at the beginning). This is accomplished by

$$\begin{aligned}
\mathcal{C}_{T_L}^{[L]} = {} & c_{1,T_{L-1}}^{[L-1]} \begin{array}{c}\triangledown \\ \overline{\mathcal{C}}_{T_L}^{[L]}\end{array} + \cdots + c_{k,T_{L-1}}^{[L-1]} \begin{array}{c}\triangledown\,\triangledown\cdots\triangledown \\ \overline{\mathcal{C}}_{T_L-k+1}^{[L]}\overline{\mathcal{C}}_{T_L-k+2}^{[L]}\cdots\overline{\mathcal{C}}_{T_L}^{[L]}\end{array} + \cdots + c_{T_L,T_{L-1}}^{[L-1]} \begin{array}{c}\triangledown\cdots\triangledown \\ \overline{\mathcal{C}}_{1}^{[L]}\cdots\overline{\mathcal{C}}_{T_L}^{[L]}\end{array} \\
& + c_{T_L+1,T_{L-1}}^{[L-1]} \begin{array}{c}\triangledown\,\triangledown\,\triangledown\cdots\triangledown \\ \mathcal{E}\overline{\mathcal{C}}_{1}^{[L]}\overline{\mathcal{C}}_{2}^{[L]}\cdots\overline{\mathcal{C}}_{T_L}^{[L]}\end{array} + \cdots + c_{T_L+m,T_{L-1}}^{[L-1]} \begin{array}{c}\triangledown\cdots\triangledown\,\triangledown\,\triangledown\cdots\triangledown \\ \underbrace{\mathcal{E}\cdots\mathcal{E}}_{m}\overline{\mathcal{C}}_{1}^{[L]}\overline{\mathcal{C}}_{2}^{[L]}\cdots\overline{\mathcal{C}}_{T_L}^{[L]}\end{array} + \cdots
\end{aligned} \tag{3.14}$$

Note that the triangular nodes above must not be counted in the OFG, since they represent (i.e., replicate) the initial cycle of the associated successor trees only.

The translation into a functional equation of the OGFs involved yields

$$\begin{aligned}
C_{T_L}^{[L]}(z) = {} & \sum_{k=0}^{T_L} c_{k,T_{L-1}}^{[L-1]} \prod_{i=T_L-k+1}^{T_L} \overline{C}_i^{[L]}(z) \\
& + \prod_{i=1}^{T_L} \overline{C}_i^{[L]}(z) \sum_{m\geq 1} c_{T_L+m,T_{L-1}}^{[L-1]}(zp_{0,L})^m.
\end{aligned} \tag{3.15}$$

Note that extending the range of summation to $k = 0$ is justified by $c_{0,T_{L-1}}^{[L-1]} = 0$.

Multiplying (3.15) by $\overline{Q}^{[L]}_{T_L}(z)$ involves another Cauchy product

$$\overline{Q}^{[L]}_{T_L}(z)C^{[L]}_{T_L}(z) = \sum_{k=0}^{T_L} c^{[L-1]}_{k,T_{L-1}}\,\overline{Q}^{[L]}_{T_L-k}(z) + \sum_{m\geq 1} c^{[L-1]}_{T_L+m,T_{L-1}}(zp_{0,L})^m.$$

Again multiplying this equation by s^{T_L} and summing up for $T_L \geq 1$, the second term above yields

$$\begin{aligned} I(s,z) &= \sum_{T_L\geq 1}\sum_{m\geq T_L+1} c^{[L-1]}_{m,T_{L-1}}(zp_{0,L})^{m-T_L}s^{T_L} \\ &= \sum_{m\geq 2} c^{[L-1]}_{m,T_{L-1}}(zp_{0,L})^m \sum_{T_L=1}^{m-1}\Big(\frac{s}{zp_{0,L}}\Big)^{T_L} \\ &= H^{[L]}(s,z), \end{aligned}$$

remember the derivation of (3.11), and we finally obtain the desired result

$$\overline{Q}^{[L]}_{T_L}(z)C^{[L]}_{T_L}(z) = [s^{T_L}]\Big(\overline{Q}^{[L]}(s,z)C^{[L-1]}_{T_{L-1}}(s) + H^{[L]}(s,z)\Big) = [s^{T_L}]Q^{[L]}(s,z) \tag{3.16}$$

according to (3.12).

Now we shift back our result in order to arrive at feasible busy periods, recall our comment prior to (3.5). Denoting the improper PGF of $\mathcal{T}_L$-feasible busy periods by

$$B^{[L]}_{T_L}(z) = B^{[L]}_{T_L,\ldots,T_1}(z) = \sum_{n\geq 0} b^{[L]}_{n,T_L} z^n, \tag{3.17}$$

where

$$b^{[L]}_{n,T_L} = \mathbf{P}\{\text{number of cycles of random } \mathcal{T}_L\text{-feasible busy period is } n\},$$

we obviously have $B^{[L]}_{T_L,\ldots,T_1}(z) = C^{[L]}_{T_L-1,\cdots,T_1-1}(z)$ and the major result of this section follows easily:

Theorem 3.1. *For $T_\ell \geq 2$, $1 \leq \ell \leq L$, the improper PGF of $\mathcal{T}_L$-feasible busy periods satisfies the recursion*

$$B^{[L]}_{T_L}(z) = \frac{[s^{T_L-2}]\dfrac{1}{zP_L(B^{[L-1]}_{T_{L-1}}(s))-s}}{[s^{T_L-1}]\dfrac{s}{B^{[L-1]}_{T_{L-1}}(s)}\cdot\dfrac{1}{zP_L(B^{[L-1]}_{T_{L-1}}(s))-s} + [s^{T_L-1}]\dfrac{s/B^{[L-1]}_{T_{L-1}}(s) - zp_{0,L}/B^{[L-1]}_{T_{L-1}}(zp_{0,L})}{s-zp_{0,L}}} \quad \textit{for } L\geq 1,$$

$$B^{[0]}(z) = z.$$

Proof. Replacing $C^{[L-1]}_{T_{L-1}}(.)$ by $B^{[L-1]}_{T_{L-1}}(.)$ in (3.16), (3.12), and (3.13) yields

$$B^{[L]}_{T_L}(z) = \frac{[s^{T_L-1}]Q^{[L]}(s,z)}{[s^{T_L-1}]Q^{[L]}(s,z)/B^{[L-1]}_{T_{L-1}}(s) - [s^{T_L-1}]H^{[L]}(s,z)/B^{[L-1]}_{T_{L-1}}(s)} \quad \text{for } L \geq 1,$$

$$B^{[0]}(z) = z$$

with

$$Q^{[L]}(s,z) = \frac{sB^{[L-1]}_{T_{L-1}}(zp_{0,L})}{zP_L\Big(B^{[L-1]}_{T_{L-1}}(s)\Big) - s} \quad \text{and} \quad H^{[L]}(s,z) = \frac{sB^{[L-1]}_{T_{L-1}}(zp_{0,L}) - zp_{0,L}B^{[L-1]}_{T_{L-1}}(s)}{zp_{0,L}-s}, \tag{3.18}$$

from where the recursion given in our theorem follows easily. □

Note that this result covers the single priority level case ($L = 1$) of [SB92] as well.

The following theorem lists a number of rather straightforward combinatorial properties:

Theorem 3.2. *The PGFs $B^{[L]}(z) = \sum_{k\geq 0} b_k^{[L]} z^k$ and $B_{T_L}^{[L]}(z) = \sum_{k\geq 0} b_{T_L,k}^{[L]} z^k$ of arbitrary and T_L-feasible busy periods, respectively, have the following properties:*

(0) *$B^{[L]}(z)$ solves the functional equation*

$$B^{[L]}(z) = zP^{[L]}\big(B^{[L]}(z)\big), \tag{3.19}$$

and $b_0^{[L]} = 0$, $b_1^{[L]} = P^{[L]}(0) = \prod_{\ell=1}^{L} p_{0,\ell} > 0$. Moreover, if $P^{[L]}(z) = \sum_{n\geq 0} p_n^{[L]} z^n$ is such that

$$\gcd(P^{[L]}) = \gcd\{n : p_n^{[L]} > 0, n \geq 1\} = d^{[L]}$$

for some integer $dL \geq 1$, then $B^{[L]}(z) = zY(z^{d^{[L]}})$ for some $Y(z) = \sum_{k\geq 0} y_k z^k$ with $\gcd(Y) = 1$.

(1) *for $n \geq 2$, $b_{n,T_L}^{[L]} \leq b_n^{[L]}$ (with the inequality being strict if some T_ℓ is finite), $b_{0,T_L}^{[L]} = b_0^{[L]} = 0$, $b_{1,T_L}^{[L]} = b_1^{[L]}$, and $\lim_{T_1,\ldots,T_L\to\infty} b_{n,T_L}^{[L]} = b_n^{[L]}$ for all n.*

(2) *$B_{T_L}^{[L]}(z)$ for $T_1, \cdots, T_L$ all being finite is a rational function.*

Proof. The functional equation for $B^{[L]}(z)$ is just a restatement of (3.4). Since a busy period obviously contains at least the initial cycle, we have $b_0^{[L]} = 0$, and the value of $b_1^{[L]} > 0$ is the probability that there are no arrivals in the initial cycle.

To show the gcd-result, we first assume $d = 1$ and employ bootstrapping on $Y(z) = P^{[L]}\big(B^{[L]}(z)\big)$: Plugging in $B^{[L]}(z) = b_1^{[L]}z + O(z^2)$ for $z \to 0$ resulting from above, it is immediately apparent that $y_k \geq p_k^{[L]}(b_1^{[L]})^k$, since all functions involved have non-negative Taylor coefficients. Hence, $y_k > 0$ when $p_k^{[L]} > 0$, which establishes $\gcd(Y) = 1$ for this case.

For $d > 1$, we have $P^{[L]}(z) = p^{[L]}(z^d)$ with $\gcd(p^{[L]}) = 1$, and it is easy to check that we can write

$$B^{[L]}(z) = z\sqrt[d]{y(z^d)}, \tag{3.20}$$

where $b^{[L]}(w) = wy(w)$ solves the functional equation

$$b^{[L]}(w) = wp^{[L]}\big(b^{[L]}(w)\big)^d. \tag{3.21}$$

Using the same reasoning as in the bootstrapping argument above, plugging in $b^{[L]}(w) = b_1^{[L]}w + O(w^2)$ into $p^{[L]}(z)$ establishes that $Y(w) = p^{[L]}\big(b^{[L]}(w)\big)$ satisfies $\gcd(Y) = 1$. Since $B^{[L]}(z) = zp^{[L]}\big(b^{[L]}(z^d)\big) = zY(z^d)$ by (3.20) and (3.21), the statement in item (0) follows for $d > 1$ as well.

Turning our attention to item (1), the upper bound holds since each feasible busy period is obviously an arbitrary one. The results for $n = 0$ resp. $n = 1$ are implied by the fact that each feasible busy period consists at least of the initial idle cycle, and that deadline restrictions apply for $n \geq 2$ only since $T_\ell \geq 2$. Finally, for some T_ℓ being finite, there are of course arbitrary busy periods of length $n \geq 2$ (occuring with probability > 0) that are not feasible, so $b_{n,T_L}^{[L]} < b_n^{[L]}$ in this case.

To prove item (2), it suffices to show that the coefficients $[s^m]Q^{[L]}(s,z)$ and $[s^m]H^{[L]}(s,z)$ —as defined in (3.18)— are rational functions (in z) for finite m, since $[s^T]f(s,z)g(s) = \sum_{m=0}^{T}[s^m]f(s,z)[w^{T-m}]g(w)$. For the first one, we have

$$\begin{aligned}
[s^m]\frac{s}{zP_L\Big(B^{[L-1]}_{\mathcal{T}_{L-1}}(s)\Big)-s} &= \sum_{k\geq 1}[s^m]\Bigg(\frac{s}{zP_L\Big(B^{[L-1]}_{\mathcal{T}_{L-1}}(s)\Big)}\Bigg)^k \\
&= \sum_{k=1}^{m}[s^{m-k}]\Bigg(\frac{1}{zP_L\Big(B^{[L-1]}_{\mathcal{T}_{L-1}}(s)\Big)}\Bigg)^k \\
&= \sum_{l=0}^{m-1}[s^l]\Bigg(\frac{1}{P_L\Big(B^{[L-1]}_{\mathcal{T}_{L-1}}(s)\Big)}\Bigg)^{m-l} z^{-(m-l)};
\end{aligned} \tag{3.22}$$

the restriction of the range of summation is justified by $P_L\Big(B^{[L-1]}_{\mathcal{T}_{L-1}}(s)\Big) = p_{0,L} > 0$, since $s/P_L\Big(B^{[L-1]}_{\mathcal{T}_{L-1}}(s)\Big) = w_1 s + w_2 s^2 + \cdots$ and $w_1 = 1/p_{0,L}$. Note that (3.22) is a polynomial of degree m in $1/z$; all coefficients are explicitly expressible in terms of $p_{k,L}$ and $b^{[L-1]}_{k,\mathcal{T}_{L-1}}$.

Second, starting from (3.11), we find

$$H^{[L]}(s,z) = \sum_{n\geq 2} b^{[L-1]}_{n,\mathcal{T}_{L-1}}(zp_{0,L})^n \sum_{k=1}^{n-1}\Big(\frac{s}{zp_{0,L}}\Big)^k = \sum_{k\geq 1}\sum_{n\geq k+1} b^{[L-1]}_{n,\mathcal{T}_{L-1}}(zp_{0,L})^{n-k}s^k,$$

which shows that

$$[s^m]H^{[L]}(s,z) = \frac{B^{[L-1]}_{\mathcal{T}_{L-1}}(zp_{0,L}) - \sum_{n=1}^{m} b^{[L-1]}_{n,\mathcal{T}_{L-1}}(zp_{0,L})^n}{(zp_{0,L})^m}. \tag{3.23}$$

Thus it follows that both (3.22) and (3.23) and hence $B^{[L]}_{\mathcal{T}_L}(z)$ are rational functions provided that $B^{[L-1]}_{\mathcal{T}_{L-1}}(z)$ is rational. Our induction argument is completed by stating the initial condition $B^{[0]}(z) = z$. □

4. General Asymptotics

Our Theorem 3.2 reveals that $B^{[L]}_{\mathcal{T}_L}(z)$ is a rational function if all $T_1,\dots,T_L$ are finite. For relatively small T_ℓ's, it is hence possible to compute $B^{[L]}_{\mathcal{T}_L}(z)$ explicitly (a powerful computer algebra system could do this), in the sense that the coefficients of the numerator and denominator polynomials are expressible in terms of $p_{k,\ell}$, recall (2.3). In practically relevant settings, however, this procedure is not feasible since service time deadlines are expressed in multiples of a cycle and therefore quite large (say, $10^3 \dots 10^6$). Therefore, it is mandatory to provide simple asymptotic formulas for $\mu^{[L]}_{\mathcal{T}_L}$ —and hence $B^{[L]}_{\mathcal{T}_L}(1)$ and $B^{[L]\prime}_{\mathcal{T}_L}(1)$, recall (2.5)— for (sufficiently) large[3] $\mathcal{T}_L$.

[3]We employ this phrase or, equivalently, $\mathcal{T}_L \to \infty$, to stress the fact that the asymptotic expression in question —i.e., the implied constant in the O-term involved— is valid provided that all $T_\ell \in \mathcal{T}_L$ satisfy $T_\ell \geq T_{\ell,0}$ for some fixed $T_{\ell,0}$; note that $T_{\ell,0}$ is usually not made explicit. Recall that one writes $f(x) = O(g(x))$ for $x \to x_0$ if there is some real constant $M > 0$ independent of x which guarantees $|f(x)| \leq M|g(x)|$ for all x in a suitable neighborhood of x^* (for $x^* = \infty$, a suitable neighborhood means $x > x_0$ for some x_0). Besides, we use the notation $f(x) = o(g(x))$ for $x \to x_0$ if $\lim_{x\to x_0} f(x)/g(x) = 0$, and $f(x) \sim g(x)$ for $x \to x_0$ if $\lim_{x\to x_0} f(x)/g(x) = 1$.

The result for $B_{\mathcal{T}_L}^{[L]}(z)$ given in Theorem 3.1 is hence attacked by asymptotic methods relying on (bivariate) singularity analysis. Actually, it is not difficult to see that the dominant singularities are determined by the solutions of the well-known functional equation $zU(s)-s=0$. Employing techniques from complex analysis, an asymptotic expansion for $B_{\mathcal{T}_L}^{[L]}(z)$ as $\mathcal{T}_L$ get large is eventually provided, which is uniformly valid for $z \in \mathcal{D}(1,\varepsilon)$[4], ε sufficiently small. Since differentiation of such an expansion is permitted, $B_{\mathcal{T}_L}^{[L]\prime}(1)$ resp. $B_{\mathcal{T}_L}^{[L]}(1)$ and hence $\mu_{\mathcal{T}_L}^{[L]}$ are readily obtained. Finally, Theorems 3.5 and 3.7 of [DS93] are applied to derive uniform asymptotic formulas for deadline meeting probabilities and all moments summarized in our major Theorem 5.9.

Hence, a key role in our derivations is played by the solutions of the functional equation

$$F(s,z) = zU(s) - s = 0, \tag{4.1}$$

where $U(s)$ denotes an analytic function with the properties

(1) $U(s)$ has non-negative Taylor coefficients and $U(0) > 0$, $U(1) \le 1$,
(2) $U'(1) < 1$ and $U''(s) \not\equiv 0$,
(3) the radius of convergence of $U(s)$ is $R_U > 1$ and $\lim_{x \to R_U-} U(x) = +\infty$.[5]

This type of functional equation has been studied extensively (and controversially) in the literature, cf. [MM78], [Can84], [MM89] for some references. In general, it has several solutions, but one is usually only interested in a particular one (with non-negative Taylor coefficients). We, however, require some results on other solutions as well. Therefore, alternative approaches based on complex analysis must be provided — traditional ones (e.g., [MM89]) rely heavily on the *a priori* assumption of non-negative Taylor coefficients.

We start our detailed treatment with some well-known properties first, namely, that our conditions on $U(s)$ ensure that the equation

$$xU'(x) - U(x) = \sum_{n\ge1}(n-1)u_n x^n - u_0 = 0 \tag{4.2}$$

has a minimal positive solution $x = \tau$ with $0 < \tau < R_U$; note that $\tau > 1$ if $U(1) = 1$. This follows from the fact that the left-hand side of (4.2) is negative for $x = 0$ but monotonically increasing to $+\infty$ for $x \to R_U$, as can be seen e.g. from the Taylor expansion above in conjunction with property (3). Defining

$$\rho = \frac{\tau}{U(\tau)}, \tag{4.3}$$

we first observe that $\rho > 1$ since, in case of $\tau \le 1$, we have $\rho - 1 > \tau U'(\tau)/U(\tau) - 1 = 0$ by property (2); if $\tau > 1$, the function $f(x) = x/U(x)$ is strictly monotonically increasing for $x < \tau$ and $f(1) \ge 1$. Note also that $\rho < \tau$ in case of $\tau > 1$.

Furthermore, it is easily checked that $F(\tau,\rho) = 0$, $F_s(\tau,\rho) = 0$, and $F_{ss}(\tau,\rho) > 0$, where $F_s(s,z)$ and $F_{ss}(s,z)$ denotes the first and second partial derivative w.r.t. s, respectively. This in fact gives rise to the following well-known

---

[4]We use $\mathcal{D}(z_0,R)$ to denote the open disk $\{z : |z-z_0| < R\}$ with radius R around z_0; $\overline{\mathcal{D}}(z_0,R)$ stands for the closed disk $\{z : |z-z_0| \le R\}$.

[5]Note that our following lemmas, except Lemma 4.2, remain valid even in case of an algebraic singularity providing $U(R_U) < \infty$ but $\lim_{x\to R_U-} U'(x) = +\infty$.

Lemma 4.1. *With the properties and notations (4.1)–(4.3) as stated above, the functional equation $F(s,z) = zU(s) - s = 0$ has a double-valued solution $s = \chi(z)$ for z in a neighborhood of $z = \rho$ (which is not necessarily the only one) and*

$$\chi(z) = \tau - \beta \cdot (1 - z/\rho)^{1/2} + \gamma \cdot (1 - z/\rho) + O\Big((1 - z/\rho)^{3/2}\Big) \qquad \text{for } z \to \rho$$

with

$$\beta = \sqrt{\frac{2U(\tau)}{U''(\tau)}}, \tag{4.4}$$

$$\gamma = \frac{3U'(\tau)U''(\tau) - U(\tau)U'''(\tau)}{3U''(\tau)^2} = \frac{\beta^2}{2\tau} - \delta, \tag{4.5}$$

$$\delta = \frac{U(\tau)U'''(\tau)}{3U''(\tau)^2}. \tag{4.6}$$

Proof. Since $zU(s) - s = 0$ may be written as $z = s/U(s)$ (provided that $U(s) \neq 0$, which is true at least for real $s \geq 0$), the problem of finding a solution of $F(s,z) = 0$ is equivalent to the problem of finding a functional inverse of $f(s) = s/U(s)$. It is easy to verify that

$$f(\tau) = \rho,$$
$$f'(\tau) = \frac{U(\tau) - \tau U'(\tau)}{U(\tau)^2} = 0,$$
$$f''(\tau) = -\frac{\tau U''(\tau)}{U(\tau)^2} = -\frac{\rho U''(\tau)}{U(\tau)} \neq 0,$$
$$f'''(\tau) = \frac{3U''(\tau) + \tau U'''(\tau)}{U(\tau)^2} = \frac{3\rho U'(\tau)U''(\tau) - \rho U(\tau)U'''(\tau)}{U(\tau)^2},$$

revealing that $f(s)$ has a double ρ-point at $s = \tau$. Thus we may apply Lagrange's inversion formula for multi-valued functions (cf. [Mar65, p. 92]) to establish that the particular (and of course uniquely determined) functional inverse $s = \chi(z) = f_\tau^{[-1]}(z)$ mapping a neighborhood of $z = \rho$ to a neighborhood of $s = \tau$ is double-valued near $z = \rho$, an algebraic branch point of first order. We immediately obtain

$$\chi(z) = \tau + \sum_{n \geq 1} \frac{1}{n!} \frac{d^{n-1}}{ds^{n-1}} \psi(s)^n \Big|_{s=\tau} (z - \rho)^{n/2}$$

with $\psi(s) = \sqrt{\frac{(s-\tau)^2}{s/U(s) - \rho}}$. In the latter formula, *any* (fixed) branch of the square-root may be chosen (we use the principal one), and it does not matter when we replace $(z - \rho)^{n/2}$ by $(-\sqrt{-\rho})^n (1 - z/\rho)^{n/2}$. Evaluating the first two terms of the sum above by means of the Taylor expansion

$$\frac{-\rho(s-\tau)^2}{s/U(s) - \rho} = -\frac{2\rho}{f''(\tau)} + \frac{2\rho f'''(\tau)}{3f''(\tau)^2}(s - \tau) + O\Big((s-\tau)^2\Big) \qquad \text{for } s \to \tau,$$

some algebraic manipulations eventually establish the result of Lemma 4.1. □

Since $\chi(z)$ is double-valued near ρ, it is possible to define two branches $\zeta(z)$, $\kappa(z)$, which are single-valued and analytic in a suitable small neighborhood of any $z_0 \neq \rho$. Thus, $F(s,z) = 0$ has two single-valued, analytic solutions in such a neighborhood. However, it is important to note that this result does not imply that there are no other solutions of $F(s,z)=0$, mapping the neighborhood of $z=z_0$ to a different neighborhood $s=s_0'$, cf. [Can84] for an example. Nevertheless, the following lemma establishes that there are no further solutions, even in the case of arbitrary $z_0=\alpha$, $0<\alpha<\rho$, if we restrict ourselves to a certain domain of s.

Lemma 4.2. *Let $U(z)$ be in accordance with (4.1)-(4.3). Then for any α, $0<\alpha<\rho$ arbitrary but fixed there is some r_α, $\tau<r_\alpha<R_U$, such that $F(s,z)=zU(s)-s=0$ restricted to the closed disk $s\in\overline{\mathcal{D}}(0,r_\alpha)$ has exactly two single-valued, analytic solutions $s=\zeta(z)$ and $s=\kappa(z)$, with values lying entirely in the interior of $\overline{\mathcal{D}}(0,r_\alpha)$ for every $z\in\mathcal{D}(\alpha,\varepsilon)$, $\varepsilon>0$ sufficiently small. Moreover, $\zeta(x)$ and $\kappa(x)$ are positive real-valued for real positive $0<x<\rho$, satisfying $\zeta(x)<\tau$ and $\kappa(x)>\tau$.*

Proof. Setting $z=\alpha$, we first look for real zeroes of $\alpha U(x)-x=0$ (which is straightforward when viewed geometrically). Since $\alpha<\rho$, it is clear that $\alpha U(\tau)-\tau<\rho U(\tau)-\tau=0$ but $\alpha U(0)>0$, so there must be a zero $\zeta_\alpha<\tau$; clearly, ζ_α decreases (towards 0) as α decreases. The Taylor expansion of $\alpha U(x)-x$ at ζ_α reads

$$\alpha U(x)-x=(x-\zeta_\alpha)\Big(\alpha U'(\zeta_\alpha)-1\Big)+R_2(x,\alpha).$$

Now, because of $\alpha U'(\zeta_\alpha)<\rho U'(\tau)=1$, remember (4.2) and (4.3), we see that ζ_α is a simple zero, and that $\alpha U(x)-x<0$ for $x\in(\zeta_\alpha,\zeta_\alpha+\epsilon)$ since $R_2(x,\alpha)=O\big((x-\zeta_\alpha)^2\big)$. In addition, $U''(x)\not\equiv 0$ guarantees that $R_2(x,\alpha)$ increases faster than the linear term, causing another zero[6] of $\alpha U(x)-x$ denoted by κ_α; clearly, $\kappa_\alpha>\tau$ and κ_α increases as α decreases. Anyway, by the first mean value theorem of differential calculus, there exists some ν, $\tau\le\nu\le\kappa_\alpha$ such that

$$\alpha U'(\nu)=\frac{\alpha U(\kappa_\alpha)-\alpha U(\tau)}{\kappa_\alpha-\tau}>\frac{\kappa_\alpha-\tau}{\kappa_\alpha-\tau}=1.$$

Thus,

$$\alpha U'(\kappa_\alpha)\ge\alpha U'(\nu)>1, \tag{4.7}$$

which reveals that κ_α is a simple zero of $\alpha U(x)-x$. Note that there is no further non-negative zero for non-negative values of α.

Considering complex arguments, we will first show that no other zeroes exist within the open disk $\mathcal{D}(0,\kappa_\alpha)$ by means of a Rouché-type argument. Let $f(z)+g(z)=\alpha U(z)$ and $f(z)=z$, hence $g(z)=\alpha U(z)-z$. According to our investigations concerning real arguments above, we obtain for any z with $\zeta_\alpha<|z|=r<\kappa_\alpha$

$$|f(z)+g(z)|=\alpha|U(z)|\le\alpha U(|z|)=\alpha U(r)<r=|z|=|f(z)|,$$

which establishes that the number of zeroes of $f(z)$ and $g(z)$ are the same, i.e., 1. Note, that this inequality also ensures that no zeroes of $\alpha U(z)-z$ on $|z|=r$ exist; the analyticity of both $f(z)$ and $g(z)$ excludes poles on $|z|=r$.

[6]This statement is possibly not true if $U(s)$ has an algebraic (and not a polar) singularity on its circle of convergence and α is (considerably) smaller than ρ, remember our remark on condition (3) of (4.1). The following Rouché-argument, however, remains valid for any $r<R_U$, and so does the proof that there is no additional fixed point on $|z|=R_U$.

On $|z| = \kappa_\alpha$, we obtain a second zero of $\alpha U(z) - z$ at $z = \kappa_\alpha$ and no others: Let u_k, $k \geq 1$ be the first non-zero coefficient of $U(z)$ apart from $u_0 > 0$, and $\psi = \kappa_\alpha e^{it}$ for $0 < t < 2\pi$ but $t \neq 2\pi/k, \ldots, 2\pi(k-1)/k$, we obviously have

$$\alpha|U(\psi)| \leq \alpha|u_0 + u_k\psi^k| + \alpha \sum_{n>k} u_n \kappa_\alpha^n < \alpha U(\kappa_\alpha) = \alpha U(|\psi|) = \kappa_\alpha. \tag{4.8}$$

If $\psi = \kappa_\alpha e^{2\pi i l/k}$, $l = 1, \ldots, k-1$, it might be tha case that $\alpha U(\psi) = \kappa_\alpha$; but then $\alpha U(\psi) - \psi = 0$ is only possible for $\psi = \kappa_\alpha$. Thus, we have showed that $z = \kappa_\alpha$ is the only fixed point.

Since our simple zero κ_α is clearly an isolated one, the modulus of the next zero ν_α (if there is one within $\mathcal{D}(0, R_U)$) must fulfill $|\nu_\alpha| > \kappa_\alpha$. Hence, choosing $r_\alpha = (\kappa_\alpha + R_\alpha)/2$ with $R_\alpha = \min(R_U, |\nu_\alpha|)$ guarantees that no additional zero lies within the closed disk $\overline{\mathcal{D}}(0, r_\alpha)$; clearly, $R_U > r_\alpha > \kappa_\alpha > \tau$.

Next, we apply the implicit function theorem to prove that ζ_α and κ_α indeed imply two solutions $\zeta_\alpha(z)$ and $\kappa_\alpha(z)$ of $F(s,z) = zU(s) - s = 0$, which are single valued in the disk $z \in \mathcal{D}(\alpha, \varepsilon_\alpha)$ for some $\varepsilon_\alpha = \varepsilon(r_\alpha) > 0$; the latter must be chosen sufficiently small in order to guarantee that $\zeta_\alpha(z)$ and $\kappa_\alpha(z)$ remain in $\overline{\mathcal{D}}(0, r_\alpha)$ for $z \in \mathcal{D}(\alpha, \varepsilon_\alpha)$. Now, since $F(s,z)$ is analytic for $s \in \overline{\mathcal{D}}(0, r_\alpha)$ and $z \in \mathcal{D}(\alpha, \varepsilon_\alpha)$, we only have to check whether

$$\left.\frac{\partial F(s,z)}{\partial s}\right|_{\substack{z=\alpha,\\ s=s_0}} = \alpha U'(s_0) - 1 \neq 0$$

for $s_0 = \zeta_\alpha$ and $s_0 = \kappa_\alpha$, respectively. This, however, is an obvious consequence of ζ_α, κ_α being simple zeroes, as previously established.

Furthermore, if we restrict α to the closed interval $I = [\alpha_l, \alpha_u] \subset (0, \rho)$, it is not hard to show by topological means that there is some $\varepsilon > 0$ independent of α such that $\mathcal{D}(\alpha, \varepsilon) \subseteq \mathcal{D}(\alpha, \varepsilon_\alpha)$ for any $\alpha \in I$. For, assuming the contrary, i.e. that for every $\delta > 0$ there is some $\alpha_\delta \in I$ such that $\varepsilon_{\alpha_\delta} < \delta$, we can define an infinite sequence $\{\alpha^n\}_{n \geq 1}$ with $\alpha^n = \alpha_{\delta_0/n}$ for some $\delta_0 > 0$. Compactness of I ensures that there is an infinite convergent subsequence $\{\alpha^{n_i}\}_{i \geq 1}$, having a limit $\alpha \in I$; this limit α must fulfill $\varepsilon_\alpha = 0$, establishing the required contradiction.

As a final step, we will remove the dependence on α and show that $\zeta_\alpha(z) = \zeta(z)$ and $\kappa_\alpha(z) = \kappa(z)$ is independent of α. This is easily done by a straightforward analytic continuation argument: We start with some $\alpha_u < \rho$ (but close to ρ), so that $\zeta_{\alpha_u}(z), \kappa_{\alpha_u}(z)$ must coincide with the two analytic branches $\zeta(z), \kappa(z)$ of $\chi(z)$ from Lemma 4.1. Our previous considerations on the regions of validity ensure that it is possible to choose a finite sequence $\alpha_u > \alpha_1 > \alpha_2 \cdots > \alpha_n = \alpha_l$ with the property

$$\alpha_u \in \mathcal{D}(\alpha_1, \varepsilon), \alpha_1 \in \mathcal{D}(\alpha_2, \varepsilon), \cdots, \alpha_{n-1} \in \mathcal{D}(\alpha_n, \varepsilon).$$

Thus, the regions of validity of z as established by a repeated application of our previous results form a finite open covering of a set containing the interval $[\alpha_l, \alpha_u]$, which establishes that the functions $\zeta_{\alpha_i}(z)$ and $\kappa_{\alpha_i}(z)$ must conincide for all i. This completes the proof of Lemma 4.2. □

The next lemma is analogous to Lemma 4.2 for $\alpha > \rho$.

Lemma 4.3. *Let $U(s)$ be in accordance with (4.1)–(4.3). Then there exists some $\nu > 0$ and some r, $\tau < r < R_U$ such that, for any α with $\rho < \alpha < \rho + \nu$ arbitrary but fixed, $F(s,z) = zU(s) - s = 0$ has exactly two single-valued, analytic solutions $s = \zeta(z)$ and $s = \kappa(z)$ (which are of course the same for all α), which lie entirely in the interior of the* *closed disk $s \in \overline{\mathcal{D}}(0,r)$ for every $z \in \mathcal{D}(\alpha,\varepsilon)$, $\varepsilon > 0$ sufficiently small. Moreover, $\zeta(z)$ and $\kappa(z)$ are complex-valued and conjugated for real positive z.*

Proof. Let $z = \rho$ and $s = \tau' \neq \tau$ but $|\tau'| = \tau$, then we cannot have $F(\tau',\rho) = 0$ by the same argument as used in the proof of Lemma 4.2, see (4.8). Thus, on $|s| = \tau$ there is only a (double) zero $s = \tau$ of $F(s,\rho) = 0$. Consequently, since zeroes are isolated, we can find some $r > \tau$ such that $F(s,\rho) \neq 0$ for $|s| = r$. Moreover, by continuity w.r.t. z, there is some $\nu > 0$ such that $F(s,z) \neq 0$ for $|s| = r$ and $\rho \le z \le \rho + \nu$ as well. The function

$$N(z) = \frac{1}{2\pi i}\int_{|s|=r} \frac{F_s(s,z)}{F(s,z)}\,ds = \#Z - \#P$$

counts, since $F(s,z)$ does not have poles, the number of zeroes within $|s| = r$ and is a *continuous* function in z. Since Lemma 4.2 shows that $N(z) = 2$ for $z < \rho$, this continuity implies that $N(z) = 2$ for $z \le \rho + \nu$ too.

Thus, we (again!) only have to consider the two branches $\zeta(z)$ and $\kappa(z)$ of the double-valued function $\chi(z)$, which are single-valued and analytic for $z > \rho$. First of all, since $\zeta(x)$ decreases as x decreases according to the proof of Lemma 4.2, it is clear that $\zeta(z)$ must be the solution involving the principal branch of $w^{1/2} = (1 - z/\rho)^{1/2}$ in the expansion of Lemma 4.1. Therefore,

$$\begin{aligned}\zeta(z) &= \tau - \beta\sqrt{1 - z/\rho} + \gamma\cdot(1 - z/\rho) + O\big((1 - z/\rho)^{3/2}\big),\\ \kappa(z) &= \tau + \beta\sqrt{1 - z/\rho} + \gamma\cdot(1 - z/\rho) + O\big((1 - z/\rho)^{3/2}\big)\end{aligned} \tag{4.9}$$

for $-\pi/2 \le \arg(1 - z/\rho) < 3\pi/2$, where the principal branch of the square-root for $-\pi/2 \le w < 3\pi/2$ is to be used. Note that "slicing" along the negative imaginary axis $\arg w = -\pi/2$ provides a domain of validity for z excluding the positive imaginary "axis" above $z = \rho$ only, thus covering both real $z < \rho$ and $z > \rho$. For the latter case, we substitute $1 - z/\rho = (z/\rho - 1)e^{i\pi}$ to obtain

$$\begin{aligned}\zeta(z) &= \tau - i\beta\sqrt{z/\rho - 1} - \gamma\cdot(z/\rho - 1) + O\big((1 - z/\rho)^{3/2}\big),\\ \kappa(z) &= \tau + i\beta\sqrt{z/\rho - 1} - \gamma\cdot(z/\rho - 1) + O\big((1 - z/\rho)^{3/2}\big).\end{aligned} \tag{4.10}$$

The latter expansions reveal that both solutions are indeed complex valued for real $z > \rho$. That they are conjugated follows easily from the fact that, given a solution $s = f(z)$ of $zU(s) - s = 0$, another one is obtained by $g(z) = \overline{f(\overline{z})}$. Since $f(z) \neq g(z)$ due to $f(z)$ being complex-valued for positive z, they must coincide with $\zeta(z)$ and $\kappa(z)$ and our assertion follows. □

Summarizing the results of Lemma 4.2, Lemma 4.3, and Lemma 4.1, we can conclude that for $0 < \alpha < \rho + v$, $v > 0$ sufficiently small, there is some r_α with $\tau < r_\alpha < R_U$ such that $F(s,z) = 0$ has exactly two solutions $\zeta(z)$, $\kappa(z)$ (formed by the two analytic branches of a single double valued solution, hence "joining" at $\alpha = \rho$), which lie entirely in $\overline{\mathcal{D}}(0,r_\alpha)$ for $z \in \mathcal{D}(\alpha,\varepsilon)$ for ε sufficiently small.

It is not hard to see that $\zeta(z)$ is the well-known "natural" solution (positive Taylor coefficients, cf. [MM89]) of $zU(s) - s = 0$, which is in fact analytic in a much larger domain and fulfills $\lim_{z\to 0}\zeta(z) = 0$; our Lemma 4.2 did not provide the whole domain of analyticity. However, whereas including $\alpha_l = 0$ in the statement of Lemma 4.2 would be possible for $\zeta(z)$, this is not true for $\kappa(z)$, which is nonzero ($= R_U$) at $z = 0$ and could even tend to infinity as

$z \to 0+$. Anyway, the following lemma provides the required (well-known, cf. [MM89]) facts about $\zeta(z)$:

Lemma 4.4. *Let $U(s)$ be in accordance with (4.1)-(4.3). Then, the solution $\zeta(z)$ of $F(s,z) = zU(s) - s = 0$ as established by Lemma 4.2, Lemma 4.3 is analytic in the indented disk $\Delta_\rho = \Delta_\rho(\eta,\varphi,d) = \{z : |z| \le \rho+\eta, |\arg(z-\rho_l) - \arg(\rho_l)| \ge \varphi, z \ne \rho_l = \rho e^{2\pi i l/d}, 1 \le l \le d\}$ for some $\eta > 0$, $0 < \varphi < \pi/2$, and $d = \gcd(U) = \gcd\{n : u_n > 0, n \ge 1\}$. It has exactly d algebraic singularity of square-root type at $z = \rho_l$, $1 \le l \le d$, on its circle of convergence and satisfies*

$$\zeta(z) = \frac{\rho_l}{\rho}\Big[\tau - \beta\cdot(1-z/\rho_l)^{1/2} + \gamma\cdot(1-z/\rho_l) + O\Big((1-z/\rho_l)^{3/2}\Big)\Big] \qquad \text{for } z \to \rho_l \text{ in } \Delta_\rho. \tag{4.11}$$

The Taylor expansion $\zeta(z) = \sum_{n\ge1}\zeta_n z^n$ has non-negative coefficients, $\zeta_1 = U(0) > 0$, and $\zeta_m > 0$ for some $m \ge 2$. Finally, $\zeta(0) = 0$ but $\zeta(z) \ne 0$ for all $z \ne 0$.

Proof. From Lemma 4.2 it follows that $\zeta(z)$ is analytic in $\mathcal{D}(\alpha,\varepsilon)$ for any $0 < \alpha < \rho$. Remembering the appropriate proof, we know that $\zeta(x) \to 0$ for real positive $x \to 0+$ and that $F_s(0,0) = -1 \ne 0$; hence, $\zeta(z)$ may in fact be continued analytically to $\mathcal{D}(\alpha,\varepsilon)$ for $0 \le \alpha < \rho$ according to the implicit function theorem. This guarantees that the Taylor expansion of $\zeta(z)$ around $z = 0$ exists. That all coefficients are positive follows easily from Lagrange's inversion formula (cf. [Mar65, p. 86]), yielding

$$\zeta_n = \frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}U^n(x)\Big|_{x=0}, \tag{4.12}$$

since all coefficients of $U(z)$ and hence in $U(z)^n$ are non-negative. We also assumed that $U(0) > 0$ and $U''(z) \not\equiv 0$, which shows that $\zeta_1 = U(0) > 0$ and also $\zeta_m > 0$ for some $m \ge 2$. Finally, since $\gcd(U) = d$, it follows that we can write $U(z) = u(z^d)$. Plugging this into (4.12), it is apparent that $\zeta_n \ne 0$, $n \ge 2$, is only possible for $n-1 = k\cdot d$ since $\frac{d^{n-1}}{dx^{n-1}}U^n(x) = dn\frac{d^{n-2}}{dx^{n-2}}u^{n-1}(x^d)u'(x^d)x^{d-1}$ evaluated at $x = 0$ can only be non-zero when d divides $n-1$. Hence, $\zeta(z)$ may be written as $\zeta(z) = z\Upsilon(z^d)$ with $\gcd(\Upsilon) = 1$. Note that all those properties have already been derived by purely combinatorial arguments in item (0) of Theorem 3.2.

Since all coefficients are non-negative, it is a trivial consequence of Pringsheim's theorem and the result of our Lemma 4.2 that $z = \rho$ is the singularity determining the radius of convergence. By (4.9) it is clear that it is an algebraic singularity of square-root type. Setting $w = ze^{2\pi i l/d}$ so that $w \to \rho_l$ when $z \to \rho$, we observe

$$\begin{aligned}\frac{\zeta(w)}{w} &= \Upsilon(w^d) = \Upsilon(z^d) = \frac{\zeta(z)}{z} = \\ &= \frac{\tau}{\rho} - \frac{\beta}{\rho}(1-z/\rho)^{1/2} + \gamma\cdot(1-z/\rho) + O\Big((1-z/\rho)^{3/2}\Big) \qquad \text{for } z \to \rho = \rho_0 \\ &= \frac{\tau}{\rho} - \frac{\beta}{\rho}(1-w/\rho_l)^{1/2} + \gamma\cdot(1-w/\rho_l) + O\Big((1-w/\rho_l)^{3/2}\Big) \qquad \text{for } w \to \rho_l,\end{aligned}$$

from where the expansion given in our lemma follows immediately. Hence, we have algebraic singularities at ρ_l, $1 \le l \le d$, on the circle of convergence, but no others: Using the argument underlying (4.8) in the proof of Lemma 4.2, we find that for $|z_0| = \rho$ but $z_0 \ne \rho_l$

$$|\zeta(z_0)| < \zeta(|z_0|) = \zeta(\rho) = \tau$$

and hence $\left|z_0 U'\big(\zeta(z_0)\big)\right| < \rho U'(\tau) = 1$, which shows that z_0 is a regular point according to the implicit function theorem. This permits us to extend the region of analyticity to the indented disk Δ_ρ defined in Lemma 4.4.

What remains to be done is to show that $\zeta(z) \neq 0$ for $z \neq 0$; assuming the contrary, i.e. , $\zeta(z_0) = 0$ for $z_0 \neq 0$, the defining equation implies $0 = \zeta(z_0) = z_0 U\big(\zeta(z_0)\big) = z_0 U(0) \neq 0$, providing the required contradiction. This completes the proof of Lemma 4.4. □

The next three lemmas provide a uniform asymptotic expansion for $g_n(z) = [s^n]G(s,z)$ for $n \to \infty$, $z \in \mathcal{D}(\alpha, \varepsilon)$, where $G(s,z)$ denotes a function analytic in a neighborhood of $s = 0$ and $z = \alpha$. Note that —by virtue of well-known theorems from the theory of analytic functions of two complex variables, cf. [Mar65, p. 101ff]— $g_n(z)$ is *analytic* in a neighborhood of $z = \alpha$; we will use this important fact frequently and without explicit notice.

The following lemma deals with the most important case where both solutions of our functional equation exist. For its proof, we use some well-known asymptotic techniques relying on singularity analysis (extended to bivariate functions, as in [SB94], for example). Such techniques exploit the fact that the Taylor coefficients of analytic functions are primarily determined by the singularities on the circle of convergence. An overview of asymptotic methods, in particular of translation lemmas and the method of Darboux, may be found in [FO90] and [Ben74]. However, we will only need elementary techniques, namely subtracted singularities and Cauchy's estimates.

Lemma 4.5. *Let $0 < \alpha < \infty$ be arbitrary but fixed. Suppose that $U(s)$ and $W(s)$ are analytic within the open disk $\mathcal{D}(0, R_U)$ and that there exists some r_α, $0 < r_\alpha < R_U$ such that*

$$G(s,z) = \frac{W(s)}{zU(s) - s}$$

has at most two simple poles $s = \zeta(z)$ and $s = \kappa(z)$, lying entirely in the interior of the closed disk $s \in \overline{\mathcal{D}}(0, r_\alpha)$ for every $z \in \mathcal{D}(\alpha, \varepsilon)$, $\varepsilon > 0$ sufficiently small. Then, $g_n(z) = [s^n]G(s,z)$ is analytic and fulfills

$$g_n(z) = \frac{W\big(\zeta(z)\big)}{1 - zU'\big(\zeta(z)\big)}\zeta(z)^{-(n+1)} + \frac{W\big(\kappa(z)\big)}{1 - zU'\big(\kappa(z)\big)}\kappa(z)^{-(n+1)} + O(r_\alpha^{-n})$$

for $n \to \infty$, where the remainder term denotes an analytic function and is uniform for $z \in \mathcal{D}(\alpha, \varepsilon)$.

Proof. Since $G(s,z)$ is clearly analytic in a neighborhood of $s = 0$ and $z = \alpha > 0$, it follows from standard devices (cf. [Mar65, Thm. 3.8]) that $g_n(z)$ is analytic in a neighborhood of $z = \alpha$. Expanding $zU(s) - s$ in powers of $s - \zeta(z)$ by using its bivariate Taylor expansion, we find

$$\begin{aligned} zU(s) - s &= \Big[zU'\big(\zeta(z)\big) - 1\Big]\big(s - \zeta(z)\big) + O\Big(z\big(s - \zeta(z)\big)^2\Big) \\ &= \Big[zU'\big(\zeta(z)\big) - 1\Big]\big(s - \zeta(z)\big)\Big(1 + O\big(s - \zeta(z)\big)\Big) \end{aligned}$$

for $s \to \zeta(z)$, cf. the proof of Lemma 4.2. Note that the remainder is uniform in z and denotes an analytic function for s in a neighborhood of $\zeta(z)$ and $z \in \mathcal{D}(\alpha, \varepsilon)$, with a zero at $s = \zeta(z)$.

Using $W(s) = W\bigl(\zeta(z)\bigr) + O\bigl(s - \zeta(z)\bigr)$ for $s \to \zeta(z)$, we find

$$\begin{aligned} G(s,z) &= \frac{W(s)}{zU(s) - s} \\ &= \frac{W\bigl(\zeta(z)\bigr) + O\bigl(s - \zeta(z)\bigr)}{\Bigl(zU'\bigl(\zeta(z)\bigr) - 1\Bigr)\bigl(s - \zeta(z)\bigr)} \Bigl(1 + O\bigl(s - \zeta(z)\bigr)\Bigr) \\ &= \frac{W\bigl(\zeta(z)\bigr)}{zU'\bigl(\zeta(z)\bigr) - 1} \cdot \frac{1}{s - \zeta(z)} + O(1) \qquad \text{for } s \to \zeta(z). \end{aligned} \tag{4.13}$$

The remainder $O(1)$ is uniform in z and denotes an analytic function

$$R_1(s,z) = G(s,z) - \frac{W\bigl(\zeta(z)\bigr)}{zU'\bigl(\zeta(z)\bigr) - 1} \cdot \frac{1}{s - \zeta(z)},$$

which has no singularity at $s = \zeta(z)$. However, since the *subtracted singularity term* has no further singularities, it is clear that $R_1(s,z)$ must still have the remaining singularities of $G(s,z)$.

Repeating the derivations above with $\kappa(z)$ instead of $\zeta(z)$, we thus obtain

$$\begin{aligned} R_1(s,z) &= G(s,z) + O(1) \\ &= \frac{W\bigl(\kappa(z)\bigr)}{zU'\bigl(\kappa(z)\bigr) - 1} \cdot \frac{1}{s - \kappa(z)} + O(1) \qquad \text{for } s \to \kappa(z). \end{aligned} \tag{4.14}$$

The uniform remainder $O(1)$ above denotes a function $R_2(s,z)$, which is obviously analytic for $s \in \overline{\mathcal{D}}(0, r_\alpha)$, $z \in \mathcal{D}(\alpha, \varepsilon)$.

The remaining thing to do is to determine the desired coefficient, which is easy due to the geometric series involved in the major terms in (4.13) and (4.14). We finally obtain

$$g_n(z) = [s^n]G(s,z) = -\frac{W\bigl(\zeta(z)\bigr)}{zU'\bigl(\zeta(z)\bigr) - 1}\zeta(z)^{-(n+1)} - \frac{W\bigl(\kappa(z)\bigr)}{zU'\bigl(\kappa(z)\bigr) - 1}\kappa(z)^{-(n+1)} + O(r_\alpha^{-n})$$

for $n \to \infty$, uniformly for $z \in \mathcal{D}(\alpha, \varepsilon)$. The remainder term follows easily from Cauchy's estimates, which imply that $[s^n]R_2(s,z) = O(r_\alpha^{-n})$. This eventually completes the proof of Lemma 4.5. □

Remembering the fact that $zU(s) - s$ has two simple zeroes for $z \neq \rho$ but a double one for $z = \rho$, Lemma 4.5 is not directly applicable in the latter case. However, since the double pole results from the simple poles $\zeta(z)$ and $\kappa(z)$ joining at $z = \rho$ (cf. the proof of Lemma 4.2), the appropriate asymptotic behavior may be obtained via expressing the major terms of the expansion provided by Lemma 4.5 in fractional powers by means of (4.10); note that $g_n(z)$ and the function represented by the remainder term (and hence the sum of the major terms!) must be analytic even at $z = \rho$.

Lemma 4.6. *With the notations and conditions of Lemma 4.5 and Lemma 4.1, in a neighborhood of $z = \rho$ we have the expansion*

$$g_n(z) = \tau^{-(n+2)}\beta W(\tau)e^{-\delta(n+1)v^2/\tau}\frac{\sin\left(\frac{\beta(n+1)v}{\tau}\right)}{v}$$
$$\cdot\left[1-\left(\frac{\beta W'(\tau)}{W(\tau)}-\frac{2\delta}{\beta}\right)v\cot\left(\tfrac{\beta(n+1)v}{\tau}\right)+O\left(v^2\cot\left(\tfrac{\beta(n+1)v}{\tau}\right)\right)\right]$$

for $n \to \infty$, which is uniformly valid for all $v = \sqrt{z/\rho - 1} = O(1/n)$, i.e., $|z-\rho| = O(1/n^2)$.

Proof. In a neighborhood of $z = \rho$ with $-1\pi/2 \le \arg(z/\rho - 1) < 3\pi/2$, we may of course substitute expansion (4.10) for $\zeta(z)$ and $\kappa(z)$ into the result of Lemma 4.5; the fact that $g_n(z)$ must be analytic even at $z = \rho$ reveals that "slicing" the domain of validity is in fact not necessary here.

We consider the first term (with k replacing $n+1$), i.e.,

$$g_k^\zeta(z) = \frac{W\big(\zeta(z)\big)}{1 - zU'\big(\zeta(z)\big)}\zeta(z)^{-k}$$

first. Abbreviating $v = \sqrt{z/\rho - 1} = O(1/n) = O(1/k)$ and remembering expansion (4.10) for $\zeta(z)$, we have

$$\begin{aligned}\zeta(z)^{-k} &= \left(\tau - i\beta v - \gamma v^2 + O(v^3)\right)^{-k}\\ &= \tau^{-k}e^{-k\log\left(1 - i\beta v/\tau - \gamma v^2/\tau + O(v^3)\right)}\\ &= \tau^{-k}e^{-k\left(-i\beta v/\tau - \gamma v^2/\tau + \beta^2v^2/(2\tau^2) + O(v^3)\right)}\\ &= \tau^{-k}e^{i\beta kv/\tau}e^{-\delta kv^2/\tau}\left(1 + O(kv^3)\right) \qquad \text{uniformly for } v = O(1/k) \text{ and } k \to \infty,\end{aligned}$$

where we used the relation between γ and δ from (4.5). Differentiating expansion (4.10) w.r.t. z, we find

$$\zeta'(z) = \frac{-i\beta}{2\rho}\cdot\frac{1}{v} - \frac{\gamma}{\rho} + O(v) \qquad \text{for } v \to 0;$$

differentiating the defining equation $zU\big(\zeta(z)\big) - \zeta(z) = 0$ w.r.t. z, we obtain $U\big(\zeta(z)\big) = \zeta'(z)\big[1 - zU'\big(\zeta(z)\big)\big]$ and hence

$$\begin{aligned}\frac{1}{1 - zU'\big(\zeta(z)\big)} &= \frac{\zeta'(z)}{U\big(\zeta(z)\big)} = \frac{z\zeta'(z)}{\zeta(z)} \qquad (4.15)\\ &= \frac{(1+v^2)\big(-i\beta/(2v) - \gamma + O(v)\big)}{\tau - i\beta v + O(v^2)}\\ &= \frac{-i\beta}{2\tau}\left(v^{-1} - \frac{2i\gamma}{\beta} + O(v)\right)\left(1 + \frac{i\beta v}{\tau} + O(v^2)\right)\\ &= \frac{-i\beta}{2\tau}\left(v^{-1} - \frac{2i\gamma}{\beta} + \frac{i\beta}{\tau} + O(v)\right) = \frac{-i\beta}{2\tau}\left(v^{-1} + \frac{2i\delta}{\beta} + O(v)\right)\end{aligned}$$

for $v \to 0$. Finally, using the Taylor expansion of $W(s)$ at $s = \tau$, we easily find

$$W\big(\zeta(z)\big) = W(\tau) - i\beta W'(\tau)v + O(v^2) \qquad \text{for } v \to 0.$$

Putting everything together, we eventually obtain

$$\begin{aligned} g_k^{\zeta}(z) &= \tau^{-k} e^{i\beta kv/\tau} e^{-\delta kv^2/\tau}\Big(1 + O(kv^3)\Big) \\ &\quad \cdot \frac{-i\beta}{2\tau}\Big(v^{-1} + \frac{2i\delta}{\beta} + O(v)\Big) \cdot W(\tau)\Big(1 - \frac{i\beta W'(\tau)}{W(\tau)}v + O(v^2)\Big) \\ &= \tau^{-k-1}\beta W(\tau) e^{-\delta kv^2/\tau} \frac{e^{i\beta kv/\tau}}{2iv}\bigg(1 - \Big(\frac{i\beta W'(\tau)}{W(\tau)} - \frac{2i\delta}{\beta}\Big)v + O(v^2)\bigg) \end{aligned}$$

for all $v = O(1/k)$ and $k \to \infty$; note that $O(kv^3) = O(v^2)$ since $kv^3 \le Mv^2$ due to $v = O(1/k)$. In order to determine the second term $g_k^{\kappa}(z)$ in the result of Lemma 4.5, we just have to replace β by $-\beta$, cf. expansions (4.10). We therefore obtain

$$g_k^{\kappa}(z) = -\tau^{-k-1}\beta W(\tau) e^{-\delta kv^2/\tau} \frac{e^{-i\beta kv/\tau}}{2iv}\bigg(1 + \Big(\frac{i\beta W'(\tau)}{W(\tau)} - \frac{2i\delta}{\beta}\Big)v + O(v^2)\bigg)$$

for all $v = O(1/k)$ and $k \to \infty$; hence it follows that

$$\begin{aligned} g_n(z) &= g_{n+1}^{\zeta}(z) + g_{n+1}^{\kappa}(z) + O(r_\alpha^{-n}) \\ &= \tau^{-(n+2)}\beta W(\tau) e^{-\delta(n+1)v^2/\tau} \\ &\quad \cdot \left[\frac{\sin\Big(\frac{\beta(n+1)v}{\tau}\Big)}{v} - \Big(\frac{\beta W'(\tau)}{W(\tau)} - \frac{2\delta}{\beta}\Big)\cos\Big(\frac{\beta(n+1)v}{\tau}\Big) + O(v)\right] + O\Big(r_\alpha^{-n}\Big). \end{aligned}$$

Noting that the remainder $O(r_\alpha^{-n})$ vanishes in the remainder already present since $r_\alpha > \tau$ according to Lemma 4.5, some straightforward algebraic manipulations finally establish the result of Lemma 4.6. □

The last lemma dealing with the coefficients of a bivariate analytic function is devoted to the case where only one solution of our functional equation exists. It is thus very similar to Lemma 4.5. However, we spend some effort on further evaluating the remainder term.

Lemma 4.7. *Let $0 < \alpha < \infty$ be arbitrary but fixed. With the notations of Lemma 4.4, suppose that $U(s)$ and $W(s)$ are analytic within the indented disk $\Delta_\varrho = \Delta_\varrho(\eta, \phi, d)$ (for the same d) and fulfill*

$$W(s) = w(\varrho_l) - x(\varrho_l)(1 - s/\varrho_l)^{1/2} + \gamma_W(\varrho_l)(1 - s/\varrho_l) + O\Big((1 - s/\varrho_l)^{3/2}\Big) \tag{4.16}$$

$$U(s) = u(\varrho_l) - v(\varrho_l)(1 - s/\varrho_l)^{1/2} + \gamma_U(\varrho_l)(1 - s/\varrho_l) + O\Big((1 - s/\varrho_l)^{3/2}\Big) \tag{4.17}$$

for $s \to \varrho_l$ in Δ_ϱ. If

$$G(s, z) = \frac{W(s)}{zU(s) - s}$$

is such that it has at most one simple pole $s = \zeta(z)$ lying entirely in the interior of the closed disk $s \in \overline{\mathcal{D}}(0,\varrho)$ for every $z \in \mathcal{D}(\alpha,\varepsilon)$, $\varepsilon > 0$ sufficiently small, then $g_n(z) = [s^n]G(s,z)$ is analytic and fulfills

$$g_n(z) = \frac{W\big(\zeta(z)\big)\zeta(z)^{-(n+1)}}{1 - zU'\big(\zeta(z)\big)} + O_1\big(R_1(z)\big)\Big[O_1\big(R_2(z)\big)n^{-3/2}\varrho^{-n} + O\big(n^{-5/2}\varrho^{-n}\big)\Big] \quad (4.18)$$

for $n \to \infty$, where

$$R_1(z) = \frac{dw(\varrho)}{\varrho - zu(\varrho)} \quad \text{and} \quad R_2(z) = \frac{-1}{2\sqrt{\pi}}\Big(\frac{x(\varrho)}{w(\varrho)} + \frac{zv(\varrho)}{\varrho - zu(\varrho)}\Big). \quad (4.19)$$

$O_1(.)$ denotes a O-term with implied constant $M \le 1 + \epsilon_O$ for some small $\epsilon_O \ge 0$; it could entirely be omitted for both $R_1(z)$ and $R_2(z)$ in case of $d = 1$. All remainder terms represent analytic functions and are uniform for $z \in \mathcal{D}(\alpha,\varepsilon)$.

Proof. The major term is derived exactly as in Lemma 4.5. To establish the remainder term, we first note that $G(s,z)$ is obviously analytic for $s \in \Delta_\varrho$ for any $z \in \mathcal{D}(\alpha,\varepsilon)$. Abbreviating $w = w(\varrho_l)$ and similarily for x, u, v, we have

$$\begin{aligned} G(s,z) &= \frac{w - x(1 - s/\varrho_l)^{1/2} + O\big((1 - s/\varrho_l)\big)}{zu - zv(1 - s/\varrho_l)^{1/2} + O\big((1 - s/\varrho_l)\big) + (1 - s/\varrho_l - 1)\varrho_l} \\ &= \frac{w}{zu - \varrho_l} \cdot \frac{1 - \frac{x}{w}(1 - s/\varrho_l)^{1/2} + O\big((1 - s/\varrho_l)\big)}{1 - \frac{zv}{zu - \varrho_l}(1 - s/\varrho_l)^{1/2} + O\big((1 - s/\varrho_l)\big)} \\ &= \frac{w}{zu - \varrho_l} - \frac{w}{zu - \varrho_l}\Big[\Big(\frac{x}{w} - \frac{zv}{zu - \varrho_l}\Big)(1 - s/\varrho_l)^{1/2} + O\big((1 - s/\varrho_l)\big)\Big] \end{aligned} \quad (4.20)$$

for $s \to \varrho_l$, $1 \le l \le d$. Note that $zu - \varrho_l \neq 0$ for $z \in \mathcal{D}(\alpha,\varepsilon)$ since we excluded poles other than $\zeta(z)$ in our conditions. Now, applying a simple (multivariate) transfer lemma (cf. [FO90], [SB94]), we immediately obtain the contribution of the singularity at $s = \varrho_l$ to $[s^n]$, namely,

$$\frac{w(\varrho_l)}{\varrho_l - zu(\varrho_l)}\Big[\Big(\frac{x(\varrho_l)}{w(\varrho_l)} + \frac{zv(\varrho_l)}{\varrho_l - zu(\varrho_l)}\Big)\frac{1}{\Gamma(-1/2)}n^{-3/2}\varrho_l^{-n} + O\big(n^{-5/2}\varrho_l^{-n}\big)\Big]. \quad (4.21)$$

Note that translating $O\big((1 - s/\varrho_l)\big)$ literally would provide only a remainder involving n^{-2}. However, since $1 - s/\varrho_l$ is analytic, we may safely "jump" to the next term $O\big((1 - s/\varrho_l)^{3/2}\big)$.

Adding up the d contributions involves adding up ϱ_l^{-n} and related sums. First, with $\zeta^l = \zeta_d^l = e^{2\pi i l/d}$, $1 \le l \le d$, denoting the d-th roots of unity, we readily obtain

$$\sum_{l=1}^{d} \zeta^{-lm} = \frac{1 - \zeta^{-dm}}{1 - \zeta^{-m}} = \begin{cases} d & \text{if } m \equiv 0 \mod d, \\ 0 & \text{otherwise,} \end{cases}$$

by applying de l'Hospital's rule in the first case. Moreover, for any analytic function $f(z) = \sum_{k\geq 0} f_k z^k$, we have

$$\sum_{l=1}^{d} \zeta^{-ln} f(z\zeta^{-l}) = \sum_{k\geq 0} f_k z^k \sum_{l=1}^{d} \zeta^{-l(n+k)} = d \sum_{\substack{k\geq 0\\ k+n\equiv 0 \bmod d}} f_k z^k = O_1\big(d \cdot f(z)\big);$$

in case of $d = 1$, the $O_1(.)$ is of course not required. Since $\varrho_l = \varrho e^{2\pi i l/d} = \varrho\zeta^l$, this means that we can simply replace any instance of ϱ_l in (4.21) by ϱ when adding up if the result is multiplied by d and put in a O_1-term. Noting that $\Gamma(-1/2) = -2\sqrt{\pi}$ eventually yields the coefficients $O_1\big(R_1(z)\big)$ and $O_1\big(R_2(z)\big)$ given in our lemma. □

Corollary 4.8. *The result of Lemma* 4.7 *is valid if $U(s)$ and $W(s)$ are not singular on $|z| = \varrho_l$.*

Proof. Dealing with the major asymptotic contribution is not altered. For the remainder term, we denote by $\overline{G}(s,z)$ the singular function dealt with in Lemma 4.7 (which may be considered as a "worst case bound"). Since $G(s,z) = G(\varrho_l, z) + O(1 - z/\varrho_l)$ for $z \to \varrho_l$ here, it follows by recalling (4.20) that

$$\frac{G(s,z) - G(\varrho_l, z)}{\overline{G}(s,z) - w(\varrho_l)/(zu(\varrho_l) - \varrho_l)} = O\big((1 - z/\varrho_l)^{1/2}\big) = o(1) \qquad \text{for } z \to \varrho_l.$$

Since the constant terms $G(\varrho_l, z)$, $w(\varrho_l)/(zu(\varrho_l) - \varrho_l)$ do not matter, a simple o-type transfer lemma from [FO90] reveals that the contribution from $z = \varrho_l$ satisfies $[s^n]G(s,z) = o\big([s^n]\overline{G}(s,z)\big) = O_1\big([s^n]\overline{G}(s,z)\big)$ for $n \to \infty$ sufficiently large. □

It should be noted, however, that the remainder provided by Corollary 4.8 is a coarse estimate, which can usually be improved considerably when the singularities of $W(s)$, $U(s)$ are available.

Observe carefully that the remainders established in Lemma 4.7 resp Corollary 4.8 are also valid for Lemma 4.5 if the conditions on $W(s)$ and $U(s)$ are satisfied. However, we cannot guarantee that there are no more significant terms arising from additional zeroes of $zU(s) - s$ for $s \in \mathcal{D}(0, \varrho) \cap \mathcal{D}(0, r_\alpha)$ in this case.

For our last general lemma, we adopt standard techniques from complex analysis ([Mar65, p. 86f]) to provide some results on the functional inverse of an convergent sequence of analytic functions:

Lemma 4.9. *Let $f(z)$ and $v_n(z)$, $n \geq 1$, be analytic for $z \in \mathcal{D}(z_0, \varepsilon)$, $\varepsilon > 0$, with $f(z)$ having a simple s_0-point at $z = z_0$ and $|v_n(z)| \leq v_n = o(1)$ uniformly as $n \to \infty$. Then, the functional inverse $f_n^{[-1]}(s)$ of $f_n(z) = f(z) - v_n(z)$ exists and is analytic in $\mathcal{D}(s_0, \delta)$ for some $\delta > 0$ independently of n. Moreover, we have*

$$f_n^{[-1]}(s) = f^{[-1]}(s) + g_n(s) \quad \textit{with} \quad g_n(s) = \frac{v_n\big(f^{[-1]}(s)\big)}{f'\big(f^{[-1]}(s)\big)} + O(v_n^2)$$

uniformly for $n \to \infty$ (further terms are available).

Proof. If an analytic function $f(z)$ has a simple (obviously isolated) s_0-point at $z = z_0$, then there exists a circle $\gamma : |z - z_0| = \rho > 0$ such that $f(z)$ has no s_0-points in $\overline{\mathcal{D}}(z_0, \rho)$ except at z_0 itself. If $\Gamma = f(\gamma)$, let $\delta > 0$ be the minimal distance between s_0 and Γ.

If $|s - s_0| < \delta$, we obviously have $|f(z) - s_0| \geq \delta > |s - s_0|$ for $z \in \gamma$ and hence, by Rouché's Theorem, it follows that $f(z) - s_0$ and $f(z) - s = f(z) - s_0 + (s_0 - s)$ have the same number of zeroes inside γ, i.e., exactly one. Thus, for any function $g(z)$ analytic on $\overline{\mathcal{D}}(z_0, \rho)$, we may write

$$g\big(f^{[-1]}(s)\big) = \frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z) - s}\,dz \qquad \text{for } s \in \mathcal{D}(s_0, \delta).$$

Note that obviously $|f(z) - s| \geq \Big||f(z) - s_0| - |s - s_0|\Big| > \eta > 0$ for $z \in \gamma$ and $s \in \mathcal{D}(s_0, \delta)$.

For n sufficiently large, it is not hard to establish that similar results hold for $f_n(z) = f(z) - v_n(z)$ as well: Considering the function $f_n(z) - f_n(z_0)$, it follows by our standard continuity argument (cf. Lemma 4.3) that it has the same number of zeroes as $f(z) - f(z_0)$ for $z \in \overline{\mathcal{D}}(z_0, \rho)$, i.e., exactly one, provided that n is sufficiently large. Thus, ρ may be chosen independently of n. Moreover, for $n > n_0$ sufficiently large with $|v_{n_0}(z)| < \delta/2$ for $z \in \overline{\mathcal{D}}(z_0, \rho)$, we find $|f_n(z) - s_0| \geq \Big||f(z) - s_0| - |v_n(z)|\Big| \geq \delta/2 > |s - s_0|$ for $z \in \gamma$ and hence

$$g\big(f_n^{[-1]}(s)\big) = \frac{1}{2\pi i}\int_\gamma g(z)\frac{f_n'(z)}{f_n(z) - s}\,dz \qquad \text{for } s \in \mathcal{D}(s_0, \delta/2).$$

Again, for $n > n_0$, $|f_n(z) - s| > \eta/2 > 0$ for $z \in \gamma$ and $s \in \mathcal{D}(s_0, \delta/2)$.

Choosing $g(z) = z$, some algebra shows

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_\gamma \frac{z f_n'(z)}{f_n(z) - s}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{z\big(f'(z) - v_n'(z)\big)}{f(z) - s - v_n(z)}\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{z\big(f'(z) - v_n'(z)\big)}{f(z) - s}\sum_{k\geq 0}\frac{v_n(z)^k}{\big(f(z) - s\big)^k}\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{z f'(z)}{f(z) - s}\,dz + \sum_{k\geq 1}\frac{1}{2\pi i}\int_\gamma \frac{z f'(z) v_n(z)^k}{\big(f(z) - s\big)^{k+1}}\,dz - \sum_{k\geq 0}\frac{1}{2\pi i}\int_\gamma \frac{z v_n'(z) v_n(z)^k}{\big(f(z) - s\big)^{k+1}}\,dz \end{aligned}$$

and, by partial integration,

$$\begin{aligned} \int_\gamma \frac{z f'(z) v_n(z)^k}{\big(f(z) - s\big)^{k+1}}\,dz &= \begin{bmatrix} \nu = z v_n(z)^k; & d\nu = v_n(z)^k + z k v_n(z)^{k-1} v_n'(z)\,dz \\ du = \frac{f'(z)}{(f(z)-s)^{k+1}}\,dz; & u = \frac{-1}{k(f(z)-s)^k} \end{bmatrix} \\ &= \int_\gamma \frac{v_n(z)^k}{k\big(f(z) - s\big)^k}\,dz + \int_\gamma \frac{z v_n'(z) v_n(z)^{k-1}}{\big(f(z) - s\big)^k}\,dz. \end{aligned}$$

It thus follows that

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_\gamma \frac{z f'(z)}{f(z) - s}\,dz + \sum_{k\geq 1}\frac{1}{2\pi i}\int_\gamma \frac{v_n(z)^k}{k\big(f(z) - s\big)^k}\,dz \\ &= f^{[-1]}(s) - \frac{1}{2\pi i}\int_\gamma \log\left(1 - \frac{v_n(z)}{f(z) - s}\right)dz. \end{aligned}$$

Evaluating

$$\frac{1}{2\pi i}\int_\gamma \frac{v_n(z)^k}{k\big(f(z)-s\big)^k}\,dz = \frac{1}{k}\,\mathrm{Res}_\gamma \frac{v_n(z)^k}{\big(f(z)-s\big)^k} = \frac{1}{k!}\cdot\frac{d^{k-1}}{dz^{k-1}}\frac{(z-\zeta)^k}{\big(f(z)-s\big)^k}v_n(z)^k\Big|_{z=\zeta=f^{[-1]}(s)}$$

(in particular for $k=1$) and using the (Cauchy) estimate

$$\left|\frac{1}{2\pi i}\int_\gamma \frac{v_n(z)^k}{k\big(f(z)-s\big)^k}\,dz\right| \le \frac{\rho|v_n(z)^k|}{k(\eta/2)^k} = O(v_n^k)$$

for finite k (in particular for $k=2$), the expansion given in Lemma 4.9 follows. □

5. Specific Asymptotics

In this section, we will apply the general tools developed in §4 to the result of Theorem 3.1. Our general approach is to "reduce" the PGF of $\mathcal{T}_L$-feasible busy periods $B^{[L]}_{\mathcal{T}_L}(z)$ to the solutions $B^{[L]}(z)$, $K^{[L]}(z)$ of the functional equation $zP^{[L]}(s)-s=0$ for arbitrary busy periods, cf. Theorem 3.2. This is accomplished by expressing the (solutions of the) "sequence" of functional equations

$$zP^{[L]}(s)-s=0 \quad\Longrightarrow\quad zP_L\big(B^{[L-1]}(s)\big)-s=0 \quad\Longrightarrow\quad zP_L\big(B^{[L-1]}_{\mathcal{T}_{L-1}}(s)\big)-s=0$$

in terms of their respective "predecessor". More specifically, the solutions $B^{[L]}(z)$, $K^{[L]}(z)$ of $zP^{[L]}(s)-s=0$ derived in Lemma 5.1 provide solutions $\zeta^{[L]}_\infty(z)$, $\kappa^{[L]}_\infty(z)$ of $zP_L\big(B^{[L-1]}(s)\big)-s=0$ established in Lemma 5.3. They are eventually used in Lemma 5.4 for developing the solutions $\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)$, $\kappa^{[L]}_{\mathcal{T}_{L-1}}(z)$ of $zP_L\big(B^{[L-1]}_{\mathcal{T}_{L-1}}(s)\big)-s=0$. Since it will be shown in Lemma 5.5 that $B^{[L]}_{\mathcal{T}_L}(z)$ is expressible in terms of $B^{[L-1]}_{\mathcal{T}_{L-1}}(z)$ and $\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)$, $\kappa^{[L]}_{\mathcal{T}_{L-1}}(z)$, it is only necessary to plug in the results of Lemma 5.4 to obtain a recursive formula for $B^{[L]}_{\mathcal{T}_L}(z)$ (Lemma 5.6 and Lemma 5.7). This recursion is solved in Lemma 5.8, which almost immediately leads to our major Theorem 5.9.

We start our detailled derivations with the following results on the PGF of arbitrary busy periods, recall Theorem 3.2:

Lemma 5.1. *The functional equation*

$$F(s,z) = zP^{[L]}(s)-s=0$$

with $P^{[L]}(s)=\prod_{\ell=1}^{L}P_\ell(s)$ and $P_\ell(s)$ defined by (2.3) conforms to (4.1)–(4.3) for some $\tau^{[L]}>1$, $\tau^{[L]}<\tau^{[L-1]}$ and $\rho^{[L]}>1$, $\rho^{[L]}<\tau^{[L]}$, and Lemma 4.1–Lemma 4.4 are applicable. Denoting the solutions $\zeta(z)$ and $\kappa(z)$ predicted by Lemma 4.2 by $B^{[L]}(z)$ and $K^{[L]}(z)$, respectively, $B^{[L]}(z)$ is the PGF of arbitrary busy periods and is analytic within the indented disk $\Delta_{\rho^{[L]}}=\Delta_{\rho^{[L]}}(\eta,\varphi,d^{[L]})$ with $d^{[L]}=\gcd(P^{[L]})$. It satisfies

$$B^{[L]}(z) = \frac{\rho_l^{[L]}}{\rho^{[L]}}\Big[\tau^{[L]}-\beta^{[L]}\cdot(1-z/\rho_l^{[L]})^{1/2}+O\Big((1-z/\rho_l^{[L]})\Big)\Big] \qquad \textit{for } z\to\rho_l^{[L]} \qquad (5.1)$$

with $\beta^{[L]}$ (and further coefficients) given by (4.4), and

$$B^{[L]}(1) = 1$$

$$B^{[L]'}(1) = \frac{1}{1 - P^{[L]'}(1)} = \frac{1}{1 - \sum_{\ell=1}^{L} P'_\ell(1)} > 0, \tag{5.2}$$

$$K^{[L]}(1) = \kappa_L > 1$$

$$K^{[L]'}(1) = \frac{\kappa_L}{1 - P^{[L]'}(\kappa_L)} = \frac{\kappa_L}{1 - \sum_{\ell=1}^{L} P'_\ell(\kappa_L)} < 0,$$

where κ_L is easily computed numerically as the minimal solution of $x = P^{[L]}(x)$ for $x > 1$.

Proof. Given the conditions on $P_\ell(z)$ in Section 2, establishing properties (1)–(3) of (4.1) is trivial. Thus, there is some $\tau^{[L]} > 1$ satisfying

$$\tau^{[L]} P^{[L]'}(\tau^{[L]}) - P^{[L]}(\tau^{[L]}) = 0 \tag{5.3}$$

and some $\rho^{[L]} > 1$ with

$$\rho^{[L]} = \frac{\tau^{[L]}}{P^{[L]}(\tau^{[L]})} < \tau^{[L]},$$

which are readily computed by solving (5.3) numerically. Evaluating the characteristic equation (4.2) for Level L, i.e.,

$$xP^{[L]'}(x) - P^{[L]}(x) = xP'_L(x)P^{[L-1]}(x) + xP_L(x)P^{[L-1]'}(x) - P_L(x)P^{[L-1]}(x)$$

at $x = \tau^{[L-1]}$, which solves the characteristic equation for Level $L-1$, we obtain a value of $\tau^{[L-1]}P'_L(\tau^{[L-1]})P^{[L-1]}(\tau^{[L-1]}) > 0$. Hence it follows that $\tau^{[L]} < \tau^{[L-1]}$, cf. the Taylor expansion in (4.2).

Applying Lemma 4.2 establishes that there are indeed two solutions $\zeta(z) = B^{[L]}(z)$ and $\kappa(z) = K^{[L]}(z)$, and Lemma 4.4 confirms that $B^{[L]}(z)$ is the "right" (analytic) solution of (3.19); the appropriate Taylor coefficients are uniquely defined by their recurrence relation. Note also that $F(1,1) = 0$ implies that either $B^{[L]}(1) = 1$ or $K^{[L]}(1) = 1$; the latter, however, is impossible since $K^{[L]}(1) > \tau^{[L]} > 1$ by Lemma 4.2. The expansion (5.1) is exactly the one of (4.11). Finally, remembering (4.15), we obtain

$$B^{[L]'}(z) = \frac{B^{[L]}(z)}{z} \cdot \frac{1}{1 - zP^{[L]'}\big(B^{[L]}(z)\big)} = \frac{P^{[L]}\big(B^{[L]}(z)\big)}{1 - zP^{[L]'}\big(B^{[L]}(z)\big)} \tag{5.4}$$

$$K^{[L]'}(z) = \frac{K^{[L]}(z)}{z} \cdot \frac{1}{1 - zP^{[L]'}\big(K^{[L]}(z)\big)} = \frac{P^{[L]}\big(K^{[L]}(z)\big)}{1 - zP^{[L]'}\big(K^{[L]}(z)\big)} \tag{5.5}$$

and the values given in Lemma 5.1 follow. Note that $K^{[L]'}(1) < 0$ is confirmed by plugging in (4.7) for $\alpha = 1$ in (5.5), and $B^{[L]'}(1) > 0$ follows from (2.4). □

The next Lemma states some straightforward relations of the PGF's of feasible busy periods and the PGF of arbitrary ones:

Lemma 5.2. *The improper PGF of $\mathcal{T}_L$-feasible busy periods $B^{[L]}_{\mathcal{T}_L}(z)$ has the following analytic properties:*

(1) $\lim_{\mathcal{T}_L \to \infty} B^{[L]}_{\mathcal{T}_L}(z) = B^{[L]}(z)$ *uniformly for* $|z| \leq r < \rho^{[L]}$, *that is,*

$$|V^{[L]}_{\mathcal{T}_L}(z)| = |B^{[L]}(z) - B^{[L]}_{\mathcal{T}_L}(z)| \leq V^{[L]}_{\mathcal{T}_L} = V^{[L]}_{\mathcal{T}_L}(r) = o(1) \qquad \text{as } \mathcal{T}_L \to \infty,$$

uniformly for $|z| \leq r$; $V^{[L]}_{\mathcal{T}_L}(z)$ *has non-negative Taylor coefficients.*

(2) $B^{[L]}_{\mathcal{T}_L}(1) \leq 1$ *and* $B^{[L]\prime}_{\mathcal{T}_L}(1) \leq B^{[L]\prime}(1)$ *(with the inequality being strict if some* T_t *is finite).*

Proof. Remembering Theorem 3.2, an application of the simple continuity lemma provided as Lemma 2.1 in [DS93] establishes the uniform limiting property in item (1); it will turn out that $V^{[L]}_{\mathcal{T}_L}$ is in fact exponentially small. Note that $B^{[L]}(z)$ is convergent even for $z = \rho^{[L]}$ since it has an algebraic singularity of square-root type according to Lemma 5.1. $|V^{[L]}_{\mathcal{T}_L}(z)| \leq V^{[L]}_{\mathcal{T}_L}(r)$ for $|z| = r$ follows easily from the non-negative coefficients of $V^{[L]}_{\mathcal{T}_L}(z)$. Finally, the values given in item (2) are obvious since $1 < \rho^{[L]}$ lies in the region of analyticity. □

We should mention that Lemma 5.2 is convenient but not really necessary. Instead of relying on the combinatorial Theorem 3.2, it would be possible to derive this result by analytically studying the solutions of our functional equations. In fact, we could state Lemma 5.2 as an (induction) hypothesis and employ an all encompassing induction proof of our major lemmas as well.

The next lemma establishes the solutions $\zeta^{[L]}_\infty(z)$, $\kappa^{[L]}_\infty(z)$ of the important functional equation $zP_L\big(B^{[L-1]}(s)\big) - s = 0$ in terms of the solutions $B^{[L]}(z)$, $K^{[L]}(z)$ of $zP^{[L]}(s) - s = 0$ provided in Lemma 5.1.

Lemma 5.3. *The functional equation*

$$zU^{[L]}(s) - s = zP_L\Big(B^{[L-1]}(s)\Big) - s = 0 \tag{5.6}$$

for $L \geq 2$ *conforms to* (4.1)–(4.3) *for*

$$\begin{aligned} \tau^{[L]}_\infty &= \rho^{[L]} P_L(\tau^{[L]}), 1 < \tau^{[L]}_\infty < \rho^{[L-1]} \\ \rho^{[L]}_\infty &= \rho^{[L]} > 1, \end{aligned} \tag{5.7}$$

where $\tau^{[L]}$ *and* $\rho^{[L]}$ *are defined in Lemma* 5.1 *and*

$$\begin{aligned} B^{[L-1]}(\tau^{[L]}_\infty) &= \tau^{[L]} \\ B^{[L-1]\prime}(\tau^{[L]}_\infty) &= \frac{P^{[L-1]}(\tau^{[L]})}{1 - \rho^{[L]} P_L(\tau^{[L]}) P^{[L-1]\prime}(\tau^{[L]})}. \end{aligned}$$

Lemma 4.1, *Lemma* 4.3, *and Lemma* 4.4 *are applicable, and even Lemma* 4.2 *applies with a limited range of validity. More specifically, depending on the value*

$$\alpha_L = \begin{cases} \frac{\tau^{[L-1]}}{P^{[L]}(\tau^{[L-1]})} & \text{if the radius of convergence of } P_L(z) \text{ fulfills } R_{P_L} > \tau^{[L-1]}, \\ 0 & \text{otherwise,} \end{cases} \tag{5.8}$$

where always $\alpha_L < \rho^{[L]}_\infty$, *we distinguish 2 cases:*

(1) *If* $\alpha_L < \alpha < \rho_\infty^{[L]} + \nu$ *for some* $\nu > 0$, *there is some* $r_{\alpha,L}$ *with* $\tau_\infty^{[L]} < r_{\alpha,L} < \rho^{[L-1]}$ *ensuring that there are two analytic solutions* $\zeta_\infty^{[L]}(z)$ *and* $\kappa_\infty^{[L]}(z)$ *(forming two branches of a double-valued solution with branch point* $z = \rho_\infty^{[L]}$*) which lie entirely in the interior of the closed disk* $\overline{\mathcal{D}}(0, r_{\alpha,L})$ *for* $z \in \mathcal{D}(\alpha,\varepsilon)$, ε *sufficiently small. For positive* x *with* $\alpha_L < x < \rho_\infty^{[L]}$, *we have* $\zeta_\infty^{[L]}(x) < \tau_\infty^{[L]}$ *and* $\kappa_\infty^{[L]}(x) > \tau_\infty^{[L]}$.

(2) *If* $0 < \alpha < \alpha_L$, *only* $\zeta_\infty^{[L]}(z)$ *remains within the closed disk* $s \in \overline{\mathcal{D}}(0, \rho^{[L-1]})$ *for* $z \in \mathcal{D}(\alpha,\varepsilon)$, ε *sufficiently small.*

Moreover, the solutions $\zeta_\infty^{[L]}(z)$ *and* $\kappa_\infty^{[L]}(z)$ *are explicitly expressible in terms of the functions* $B^{[L]}(z)$ *and* $K^{[L]}(z)$, *namely*

$$\zeta_\infty^{[L]}(z) = zP_L\big(B^{[L]}(z)\big) = \frac{B^{[L]}(z)}{P^{[L-1]}\big(B^{[L]}(z)\big)} \quad \text{for } z \in \Delta_{\rho^{[L]}}, \tag{5.9}$$

$$\kappa_\infty^{[L]}(z) = zP_L\big(K^{[L]}(z)\big) = \frac{K^{[L]}(z)}{P^{[L-1]}\big(K^{[L]}(z)\big)} \quad \text{for } z \in \mathcal{D}(\alpha,\varepsilon),\ \alpha > \alpha_L, \tag{5.10}$$

(valid for $z = \rho^{[L]}$ *as well), with the properties*

$$B^{[L-1]}\big(\zeta_\infty^{[L]}(z)\big) = B^{[L]}(z) \tag{5.11}$$

$$B^{[L-1]}\big(\kappa_\infty^{[L]}(z)\big) = K^{[L]}(z). \tag{5.12}$$

Proof. We first have to verify that properties (1)–(3) of (4.1) apply for $U^{[L]}(s)$. The first one follows trivially from the conditions on $P_\ell(z)$ according to (2.3) and the results of Lemma 5.1. Moreover, we also find

$$1 - U^{[L]\prime}(1) = 1 - P_L'(1)B^{[L-1]\prime}(1) = \frac{1 - P_1'(1) - \cdots - P_{L-1}'(1) - P_L'(1)}{1 - P_1'(1) - \cdots - P_{L-1}'(1)} > 0,$$

by using (5.2) and (2.4). Condition (2) of (2.3) in conjunction with the fact that $B^{[L-1]}(z)$ has non-negative Taylor coefficients establishes that $U^{[L]\prime\prime}(z) \not\equiv 0$. Finally, we have a radius of convergence

$$R_{U^{[L]}} = \begin{cases} r \le \rho^{[L-1]} & \text{with } \lim_{x\to R_{U^{[L]}}-} U^{[L]}(x) = +\infty \quad \text{if } R_{P_L} \le \tau^{[L-1]}, \\ \rho^{[L-1]} & \text{with } \lim_{x\to R_{U^{[L]}}-} U^{[L]\prime}(x) = +\infty \quad \text{otherwise.} \end{cases} \tag{5.13}$$

The former case follows from condition (3) on (2.3), the latter from the fact that $B^{[L-1]}(s)$ has an algebraic branch point of first order at $s = \rho^{[L-1]}$, so that

$$B^{[L-1]\prime}(s) \sim \frac{\beta^{[L-1]}}{2\rho^{[L-1]}}(1 - s/\rho^{[L-1]})^{-1/2} \qquad \text{for } s \to \rho^{[L-1]} \text{ in } \Delta_{\rho^{[L-1]}},$$

cf. expansion (4.9), and hence

$$U^{[L]\prime}(s) \sim \frac{\beta^{[L-1]}P_L'(\tau^{[L-1]})}{2\rho^{[L-1]}}(1 - s/\rho^{[L-1]})^{-1/2} \to +\infty \qquad \text{for } s \to \rho^{[L-1]} \text{ in } \Delta_{\rho^{[L-1]}}, \tag{5.14}$$

if only $P_L'(\tau^{[L-1]}) \neq 0$; this, however, is trivial since the PGF $P_L(z)$ has non-negative Taylor coefficients.

Hence it follows that Lemma 4.1, Lemma 4.3, and Lemma 4.4 are applicable, and we are assured that there is some $\tau_\infty^{[L]} > 1$ satisfying (4.2), that is,

$$\tau_\infty^{[L]} P_L'\Big(B^{[L-1]}(\tau_\infty^{[L]})\Big) B^{[L-1]\prime}(\tau_\infty^{[L]}) - P_L\Big(B^{[L-1]}(\tau_\infty^{[L]})\Big) = 0; \tag{5.15}$$

clearly, $\tau_\infty^{[L]} < R_{U^{[L]}} \leq \rho^{[L-1]}$. Setting $\tau = B^{[L-1]}(\tau_\infty^{[L]})$ and hence $\tau_\infty^{[L]} = \tau / P^{[L-1]}(\tau)$ according to the defining equation, cf. Lemma 5.1, it follows from (5.4) that

$$B^{[L-1]\prime}(\tau_\infty^{[L]}) = \frac{P^{[L-1]}(\tau)}{1 - \tau_\infty^{[L]} P^{[L-1]\prime}(\tau)}.$$

Inserting this in (5.15) yields, after some algebraic manipulations,

$$\begin{aligned} 0 &= \tau P_L'(\tau) - \Big(1 - \tau_\infty^{[L]} P^{[L-1]\prime}(\tau)\Big) P_L(\tau) \\ &= \tau P_L'(\tau) P^{[L-1]}(\tau) - P^{[L]}(\tau) + \tau P_L(\tau) P^{[L-1]\prime}(\tau) \\ &= \tau P^{[L]\prime}(\tau) - P^{[L]}(\tau), \end{aligned}$$

which is the characteristic equation (5.3). Hence, $\tau^{[L]} = \tau$ and $B^{[L-1]}(\tau_\infty^{[L]}) = \tau^{[L]}$ so that

$$\begin{aligned} \tau_\infty^{[L]} &= \frac{\tau^{[L]}}{P^{[L-1]}(\tau^{[L]})} = \rho^{[L]} P_L(\tau^{[L]}) \\ \rho_\infty^{[L]} &= \frac{\tau_\infty^{[L]}}{P_L\Big(B^{[L-1]}(\tau_\infty^{[L]})\Big)} = \frac{\tau_\infty^{[L]}}{P_L(\tau^{[L]})} = \rho^{[L]}. \end{aligned}$$

From the considerations at the beginning, it is clear that Lemma 4.2 also applies, without restrictions if $R_{P_L} \leq \tau^{[L-1]}$, and with the restriction that the second solution $\kappa(z)$ may not exist otherwise. More specifically, if the "key equation"

$$\alpha P_L\Big(B^{[L-1]}(\rho^{[L-1]})\Big) - \rho^{[L-1]} = \alpha P_L(\tau^{[L-1]}) - \rho^{[L-1]} \tag{5.16}$$

evaluates to a value larger than zero, it is guaranteed that the second solution exists for $z \in \mathcal{D}(\alpha, \varepsilon)$. For a negative value, only the single analytic solution $\zeta(z)$ remains since the first term does not become large enough to cause the second zero κ_α, cf. the footnote in the proof of Lemma 4.2 The value

$$\alpha_L = \frac{\rho^{[L-1]}}{P_L(\tau^{[L-1]})} = \frac{\tau^{[L-1]}}{P^{[L]}(\tau^{[L-1]})} \tag{5.17}$$

providing

$$\alpha_L P_L(\tau^{[L-1]}) - \rho^{[L-1]} = 0 \tag{5.18}$$

of course determines the limit concerning the unrestricted applicability of Lemma 4.2; note that

$$\rho_\infty^{[L]} = \frac{\tau_\infty^{[L]}}{U^{[L]}(\tau_\infty^{[L]})} > \frac{\rho^{[L-1]}}{U^{[L]}(\rho^{[L-1]})} = \alpha_L$$

since $x/U^{[L]}(x)$ is monotonically decreasing for $x \geq \tau_\infty^{[L]}$ and $\tau_\infty^{[L]} < \rho^{[L-1]}$.

However, our tools are no longer applicable for $\alpha = \alpha_L$ since we leave the domain of analyticity of $F(s, z)$. However, Lemma 4.2 establishes that the 2nd solution $\kappa(x)$ exists for real $x \geq \alpha_L$ but not for smaller positive arguments. Limiting considerations based on (4.15)

—for $\kappa(z)$ instead of $\zeta(z)$— reveal that $\lim_{x\to\alpha_L+}\kappa'(x)=0$ due to (5.14). Anyway, we do not bother ourselves with the analysis of this special case.

What remains to be done is to justify the explicit expressions for $\zeta_\infty^{[L]}(z)$ and $\kappa_\infty^{[L]}(z)$ and the properties stated in Lemma 5.3. Plugging

$$\zeta_\infty^{[L]}(z) = zP_L\big(B^{[L]}(z)\big) = \frac{zP^{[L]}\big(B^{[L]}(z)\big)}{P^{[L-1]}\big(B^{[L]}(z)\big)} = \frac{B^{[L]}(z)}{P^{[L-1]}\big(B^{[L]}(z)\big)}$$

for $z\in\Delta_{\rho^{[L]}}$ into the defining functional equation (5.6) shows

$$zP_L\Big(B^{[L-1]}\big(\zeta_\infty^{[L]}(z)\big)\Big) - \zeta_\infty^{[L]}(z) = zP_L\Bigg(B^{[L-1]}\bigg(\frac{B^{[L]}(z)}{P^{[L-1]}\big(B^{[L]}(z)\big)}\bigg)\Bigg) - zP_L\big(B^{[L]}(z)\big) \tag{5.19}$$

$$= zP_L\big(B^{[L]}(z)\big) - zP_L\big(B^{[L]}(z)\big) = 0, \tag{5.20}$$

confirming (5.9) and (5.11), However, in order not to be only formally valid, it must be ensured that $|B^{[L]}(z)|\le\tau^{[L-1]}$, but this follows trivially from the fact that $|\zeta_\infty^{[L]}(z)|<r_{\alpha,L}<R_{U^{[L]}}\le\rho^{[L-1]}$ by Lemma 4.2 and (5.13).

An analogous derivation confirms (5.10) and (5.12) for $\kappa_\infty^{[L]}(z)$. However, the restriction $\alpha>\alpha_L$ is required here to ensure that $|K^{[L]}(z)|\le\tau^{[L-1]}$; note that (5.17) implies $K^{[L]}(\alpha_L)=\tau^{[L-1]}$. This eventually completes the proof of Lemma 5.3. □

Note that Lemma 5.3 "degenerates" to Lemma 5.1 for $L=1$ since $B^{[0]}(z)=z$ according to Theorem 3.1. Actually, Lemma 5.3 is valid even for $L=1$ if we use the (natural) convention $P^{[0]}(z)\equiv 1$ (and $\tau^{[0]}=\infty$); clearly, $\zeta_\infty^{(1)}(z)=B^{(1)}(z)$ and $\kappa_\infty^{(1)}(z)=K^{(1)}(z)$.

The following major lemma provides the solutions $\zeta_{\mathcal{T}_{L-1}}^{[L]}(z)$, $\kappa_{\mathcal{T}_{L-1}}^{[L]}(z)$ of the functional equation $zP_L\big(B_{\mathcal{T}_{L-1}}^{[L-1]}(s)\big)-s=0$ in terms of the solutions $\zeta_\infty^{[L]}(z)$, $\kappa_\infty^{[L]}(z)$ of $zP_L\big(B^{[L-1]}(s)\big)-s=0$ supplied by Lemma 5.3.

Lemma 5.4. *If $B_{\mathcal{T}_{L-1}}^{[L-1]}(z)$ denotes the improper PGF of $\mathcal{T}_{L-1}$-feasible busy periods (for higher priority tasks), the functional equation*

$$F_{\mathcal{T}_{L-1}}^{[L]}(s,z) = zU_{\mathcal{T}_{L-1}}^{[L]}(s) - s = zP_L\big(B_{\mathcal{T}_{L-1}}^{[L-1]}(s)\big) - s = 0 \tag{5.21}$$

for $L\ge 2$ conforms to (4.1)–(4.3) for

$$\tau_{\mathcal{T}_{L-1}}^{[L]} \ge \tau_\infty^{[L]} > 1,\ \tau_{\mathcal{T}_{L-1}}^{[L]} - \tau_\infty^{[L]} = O\big(V_{\mathcal{T}_{L-1}}^{[L-1]}\big) \quad \text{for } \mathcal{T}_{L-1}\to\infty,$$
$$\rho_{\mathcal{T}_{L-1}}^{[L]} \ge \rho_\infty^{[L]} > 1,\ \rho_{\mathcal{T}_{L-1}}^{[L]} - \rho_\infty^{[L]} = O\big(V_{\mathcal{T}_{L-1}}^{[L-1]}\big) \quad \text{for } \mathcal{T}_{L-1}\to\infty,$$

where $\tau_\infty^{[L]}=\rho^{[L]}P_L(\tau^{[L]})$, $\rho_\infty^{[L]}=\rho^{[L]}$, and $V_{\mathcal{T}_{L-1}}^{[L-1]}$ have been defined in Lemma 5.3 and Lemma 5.2. Lemma 4.1, Lemma 4.3, and Lemma 4.4 are applicable, and Lemma 4.2 applies in the same sense as in Lemma 5.3 (however, with no restriction on the range of validity if all $T_\ell\in\mathcal{T}_{L-1}$ are finite): Depending on

$$\alpha_L = \begin{cases} \frac{\tau^{[L-1]}}{P^{[L]}(\tau^{[L-1]})} & \text{if the radius of convergence of } P_L(z) \text{ fulfills } R_{P_L}>\tau^{[L-1]}, \\ 0 & \text{otherwise,} \end{cases}$$

where always $\alpha_L<\rho^{[L]}\le\rho_{\mathcal{T}_{L-1}}^{[L]}$, we distinguish 3 cases:

(1) *If $\mathcal{T}_{L-1}$ is sufficiently large and $\alpha_L < \alpha < \rho^{[L]}_{\mathcal{T}_{L-1}} + \nu$ for some $\nu > 0$, there is some $r_{\alpha,L}$ with $\rho^{[L-1]} > r_{\alpha,L} > \tau^{[L]}_{\mathcal{T}_{L-1}} \geq \tau^{[L]}_{\infty}$ ensuring that $F^{[L]}_{\mathcal{T}_{L-1}}(s,z) = 0$ has exactly two single-valued, analytic solutions $\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)$ and $\kappa^{[L]}_{\mathcal{T}_{L-1}}(z)$ (forming two branches of a double-valued solution with branch point $z = \rho^{[L]}_{\mathcal{T}_{L-1}}$) which lie entirely in the interior of the closed disk $\overline{\mathcal{D}}(0, r_{\alpha,L})$ for every $z \in \mathcal{D}(\alpha,\varepsilon)$, $\varepsilon > 0$ sufficiently small. Note that ν, $r_{\alpha,L}$ and ε may be chosen independently of $\mathcal{T}_{L-1}$. With $\zeta^{[L]}_{\infty}(z) = zP_L\big(B^{[L]}(z)\big)$ and $\kappa^{[L]}_{\infty}(z) = zP_L\big(K^{[L]}(z)\big)$, we have the asymptotic expressions*

$$\zeta^{[L]}_{\mathcal{T}_{L-1}}(z) = \zeta^{[L]}_{\infty}(z) - \frac{z^2\zeta^{[L]\prime}_{\infty}(z)}{\zeta^{[L]}_{\infty}(z)} P'_L\Big(B^{[L-1]}\big(\zeta^{[L]}_{\infty}(z)\big)\Big) V^{[L-1]}_{\mathcal{T}_{L-1}}\big(\zeta^{[L]}_{\infty}(z)\big) + O\Big({V^{[L-1]}_{\zeta,\mathcal{T}_{L-1}}}^2\Big) \tag{5.22}$$

$$\kappa^{[L]}_{\mathcal{T}_{L-1}}(z) = \kappa^{[L]}_{\infty}(z) - \frac{z^2\kappa^{[L]\prime}_{\infty}(z)}{\kappa^{[L]}_{\infty}(z)} P'_L\Big(B^{[L-1]}\big(\kappa^{[L]}_{\infty}(z)\big)\Big) V^{[L-1]}_{\mathcal{T}_{L-1}}\big(\kappa^{[L]}_{\infty}(z)\big) + O\Big({V^{[L-1]}_{\kappa,\mathcal{T}_{L-1}}}^2\Big), \tag{5.23}$$

uniformly for $z \in \mathcal{D}(\alpha,\varepsilon)$ as $\mathcal{T}_{L-1} \to \infty$, where $V^{[L-1]}_{\zeta,\mathcal{T}_{L-1}} = V^{[L-1]}_{\mathcal{T}_{L-1}}\big(\zeta^{[L]}_{\infty}(\alpha) + \epsilon\big)$ for some $\epsilon = \epsilon(\varepsilon) > 0$ (and analogous for $V^{[L-1]}_{\kappa,\mathcal{T}_{L-1}}$). Moreover, $\zeta^{[L]}_{\mathcal{T}_{L-1}}(x) < \tau^{[L]}_{\mathcal{T}_{L-1}}$ and $\kappa^{[L]}_{\mathcal{T}_{L-1}}(x) > \tau^{[L]}_{\mathcal{T}_{L-1}}$ for positive x with $\alpha_L < x < \rho^{[L]}_{\mathcal{T}_{L-1}}$.

(2) *If all $\mathcal{T}_{L-1}$ are finite, we have the same basic result as in item (1) even for $0 < \alpha \leq \alpha_L$. However, $r_{\alpha,L}$ and ε are no longer independent of $\mathcal{T}_{L-1}$. That is, there is some $r_{\alpha,L,\mathcal{T}_{L-1}} > \tau^{[L]}_{\mathcal{T}_{L-1}}$ ensuring that $F^{[L]}_{\mathcal{T}_{L-1}}(s,z) = 0$ has exactly two single-valued, analytic solutions $\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)$ and $\kappa^{[L]}_{\mathcal{T}_{L-1}}(z)$ which lie entirely in the interior of the closed disk $\overline{\mathcal{D}}(0, r_{\alpha,L,\mathcal{T}_{L-1}})$ for every $z \in \mathcal{D}(\alpha, \varepsilon^{[L]}_{\mathcal{T}_{L-1}})$, $\varepsilon^{[L]}_{\mathcal{T}_{L-1}} > 0$ sufficiently small.*

(3) *For arbitrary $\mathcal{T}_{L-1}$ sufficiently large (including some T_ℓ being infinite) and $0 < \alpha < \alpha_L$, $F^{[L]}_{\mathcal{T}_{L-1}}(s,z) = 0$ has exactly one single-valued, analytic solution $\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)$ with the asymptotic expansion (5.22), which lies entirely in the interior of the closed disk $\overline{\mathcal{D}}(0, r_{\alpha,L})$ for every $z \in \mathcal{D}(\alpha,\varepsilon)$, $\varepsilon > 0$ sufficiently small; $r_{\alpha,L}$ and ε are independent of $\mathcal{T}_{L-1}$. Moreover, $U^{[L]}_{\mathcal{T}_{L-1}}(s)$ has a radius of convergence of at least $\rho^{[L-1]}$, with an algebraic singularity of square-root type at $s = \rho^{[L-1]}$ if all $T_\ell \in \mathcal{T}_{L-1}$ are infinite.*

Proof. We have to verify first that $U^{[L]}_{\mathcal{T}_{L-1}}(s)$ satisfies properties (1)–(3) of (4.1). Item (1) of Theorem 3.2 in conjunction with Lemma 5.2 shows that $U^{[L]}_{\mathcal{T}_{L-1}}(1) \leq U^{[L]}(1)$ and ${U^{[L]}_{\mathcal{T}_{L-1}}}'(1) \leq U^{[L]\prime}(1)$, which means that the appropriate part of the proof of Lemma 5.3 carries over literally. As far as property (3) is concerned, we know by Theorem 3.2 that $B^{[L-1]}_{\mathcal{T}_{L-1}}(z)$ is a rational function if all $T_\ell \in \mathcal{T}_{L-1}$ are finite. Hence, $U^{[L]}_{\mathcal{T}_{L-1}}(s)$ has a polar singularity in this case. On the other hand, if all T_ℓ are infinite, we have exactly the situation of Lemma 5.3. The latter also implies satisfaction of property (3) when only some T_ℓ are infinite (provided that all $\mathcal{T}_{L-1}$ are sufficiently large), since (3)'s only purpose is to ensure that $U(s)$ gets sufficiently large to provide a solution τ of (4.2), cf. our remark on condition (3) of (2.3).

Therefore, $\tau^{[L]}_{\mathcal{T}_{L-1}}$ and $\rho^{[L]}_{\mathcal{T}_{L-1}}$ exist, and since

$$0 \leq u^{[L]}_{n,\mathcal{T}_{L-1}} = [z^n]U^{[L]}_{\mathcal{T}_{L-1}}(z) \leq u^{[L]}_n = [z^n]U^{[L]}(z) \quad \text{for } n \geq 1 \qquad \text{and} \quad u^{[L]}_{0,\mathcal{T}_{L-1}} = u^{[L]}_0, \tag{5.24}$$

which is a simple consequence of Theorem 3.2, it is immediately apparent from the Taylor expansion in (4.2) that $\tau^{[L]}_{\mathcal{T}_{L-1}} \geq \tau^{[L]}_{\infty}$. From the Taylor expansion of $U^{[L]}_{\mathcal{T}_{L-1}}(x)$ at $\tau^{[L]}_{\mathcal{T}_{L-1}}$ it follows by convexity (we have non-negative $u^{[L]}_{n,\mathcal{T}_{L-1}}$) that

$$U^{[L]}_{\mathcal{T}_{L-1}}(\tau^{[L]}_{\infty}) \geq U^{[L]}_{\mathcal{T}_{L-1}}(\tau^{[L]}_{\mathcal{T}_{L-1}}) + (\tau^{[L]}_{\infty} - \tau^{[L]}_{\mathcal{T}_{L-1}})U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\mathcal{T}_{L-1}}),$$

and from the characteristic equation (4.2) we have $U^{[L]}(\tau^{[L]}_{\infty}) = \tau^{[L]}_{\infty} U^{[L]\prime}(\tau^{[L]}_{\infty})$, so that eventually

$$\begin{aligned} 0 \leq U^{[L]}(\tau^{[L]}_{\infty}) - U^{[L]}_{\mathcal{T}_{L-1}}(\tau^{[L]}_{\infty}) \leq \\ \tau^{[L]}_{\infty}\Big(U^{[L]\prime}(\tau^{[L]}_{\infty}) - U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\mathcal{T}_{L-1}})\Big) + \Big(\tau^{[L]}_{\mathcal{T}_{L-1}} U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\mathcal{T}_{L-1}}) - U^{[L]}_{\mathcal{T}_{L-1}}(\tau^{[L]}_{\mathcal{T}_{L-1}})\Big). \end{aligned} \tag{5.25}$$

Since the second term is the characteristic equation (4.2) for $\tau^{[L]}_{\mathcal{T}_{L-1}}$, it evaluates to zero, so $U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\mathcal{T}_{L-1}}) \leq U^{[L]\prime}(\tau^{[L]}_{\infty})$ and hence

$$\rho^{[L]}_{\mathcal{T}_{L-1}} = \frac{\tau^{[L]}_{\mathcal{T}_{L-1}}}{U^{[L]}_{\mathcal{T}_{L-1}}(\tau^{[L]}_{\mathcal{T}_{L-1}})} = \frac{\tau^{[L]}_{\mathcal{T}_{L-1}}}{\tau^{[L]}_{\mathcal{T}_{L-1}} U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\mathcal{T}_{L-1}})} = \frac{1}{U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\mathcal{T}_{L-1}})} \geq \frac{1}{U^{[L]\prime}(\tau^{[L]}_{\infty})} = \rho^{[L]}_{\infty}. \tag{5.26}$$

Moreover, putting together the simple convexity results

$$U^{[L]\prime}(\tau^{[L]}_{\mathcal{T}_{L-1}}) - U^{[L]\prime}(\tau^{[L]}_{\infty}) \geq (\tau^{[L]}_{\mathcal{T}_{L-1}} - \tau^{[L]}_{\infty})U^{[L]\prime\prime}(\tau^{[L]}_{\infty}),$$

the inequality

$$U^{[L]\prime}(\tau^{[L]}_{\mathcal{T}_{L-1}}) - U^{[L]\prime}(\tau^{[L]}_{\infty}) \leq U^{[L]\prime}(\tau^{[L]}_{\mathcal{T}_{L-1}}) - U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\mathcal{T}_{L-1}}),$$

resulting from (5.25) above, and

$$U^{[L]\prime}(x) - U^{[L]}_{\mathcal{T}_{L-1}}{}'(x) = P_L'\Big(B^{[L-1]}(x)\Big)B^{[L-1]\prime}(x) - P_L'\Big(B^{[L-1]}_{\mathcal{T}_{L-1}}(x)\Big)B^{[L-1]\prime}_{\mathcal{T}_{L-1}}(x) \tag{5.27}$$

$$\begin{aligned} &= B^{[L-1]\prime}(x)\Big(P_L'\Big(B^{[L-1]}(x)\Big) - P_L'\Big(B^{[L-1]}(x) - V^{[L-1]}_{\mathcal{T}_{L-1}}(x)\Big)\Big) \\ &\quad + P_L'\Big(B^{[L-1]}_{\mathcal{T}_{L-1}}(x)\Big)V^{[L-1]\prime}_{\mathcal{T}_{L-1}}(x) \end{aligned} \tag{5.28}$$

$$\begin{aligned} &\leq B^{[L-1]\prime}(x)P_L''\Big(B^{[L-1]}(x)\Big)V^{[L-1]}_{\mathcal{T}_{L-1}}(x) + P_L'\Big(B^{[L-1]}_{\mathcal{T}_{L-1}}(x)\Big)V^{[L-1]\prime}_{\mathcal{T}_{L-1}}(x) \\ &= O\Big(V^{[L-1]}_{\mathcal{T}_{L-1}}(r)\Big) \qquad \text{for } \mathcal{T}_{L-1} \to \infty, \end{aligned} \tag{5.29}$$

for any $x \geq 0$, where we used Cauchy's estimate for the derivative of $V^{[L-1]}_{\mathcal{T}_{L-1}}(z)$, we obtain

$$\tau^{[L]}_{\mathcal{T}_{L-1}} - \tau^{[L]}_{\infty} \leq O\Big(V^{[L-1]}_{\mathcal{T}_{L-1}}(r)\Big) \quad \text{as } \mathcal{T}_{L-1} \to \infty$$

for some r with $\tau^{[L]}_{\mathcal{T}_{L-1}} < r < \rho^{[L-1]}$; by virtue of (5.7), the latter inequality is satisfyable at least for $\mathcal{T}_{L-1}$ sufficiently large. This result immediately carries over to $\rho^{[L]}_{\mathcal{T}_{L-1}}$ since, by (5.26),

$$\rho^{[L]}_{\mathcal{T}_{L-1}} - \rho^{[L]}_{\infty} = \frac{U^{[L]\prime}(\tau^{[L]}_{\infty}) - U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\mathcal{T}_{L-1}})}{U^{[L]\prime}(\tau^{[L]}_{\infty})U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\mathcal{T}_{L-1}})} \leq \frac{U^{[L]\prime}(\tau^{[L]}_{\infty}) - U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\infty})}{U^{[L]\prime}(\tau^{[L]}_{\infty})U^{[L]}_{\mathcal{T}_{L-1}}{}'(\tau^{[L]}_{\mathcal{T}_{L-1}})} = O\Big(V^{[L-1]}_{\mathcal{T}_{L-1}}(r)\Big)$$

for $\mathcal{T}_{L-1} \to \infty$, where we used (5.29) for $x = \tau^{[L]}_{\infty}$.

Therefore, Lemma 4.1, Lemma 4.3, and Lemma 4.4 establishing the properties of the solutions of $F^{[L]}_{\mathcal{T}_{L-1}}(s,z)=0$ apply; Lemma 4.2 applies as well, without restrictions in case (1) and (2), and with the restriction that $\kappa^{[L]}_{\mathcal{T}_{L-1}}(z)$ does not exist in case (3). Note that $\alpha_L < \rho^{[L]}_{\mathcal{T}_{L-1}}$ since $\alpha_L < \rho^{[L]}_{\infty}$ by Lemma 5.3.

In case (1), we can remove the restriction that ν, $r_{\alpha,L}$ and ε depend on $\mathcal{T}_{L-1}$. By Lemma 5.2 we know that $B^{[L-1]}_{\mathcal{T}_{L-1}}(z) \to B^{[L-1]}(z)$ as $\mathcal{T}_{L-1}\to\infty$, uniformly for $z \in \mathcal{D}(0,\rho^{[L-1]})$. Hence it is clear that the "limits" $\lim_{\mathcal{T}_{L-1}\to\infty}\nu^{[L]}_{\mathcal{T}_{L-1}}$, $\lim_{\mathcal{T}_{L-1}\to\infty} r_{\alpha,L,\mathcal{T}_{L-1}}$, and $\lim_{\mathcal{T}_{L-1}\to\infty}\varepsilon^{[L]}_{\mathcal{T}_{L-1}}$ provided by Lemma 4.2 and Lemma 4.3 applied to $zU^{[L]}_{\mathcal{T}_{L-1}}(s)-s=0$ as $\mathcal{T}_{L-1}\to\infty$ are $\nu^{[L]}_{\infty}$, $r_{\alpha,L,\infty}$ and $\varepsilon^{[L]}_{\infty}$ for $zU^{[L]}(s)-s=0$, cf. Lemma 5.3. It follows that choosing $\nu=\nu^{[L]}_{\infty}/2$ is admissible provided that $\mathcal{T}_{L-1}$ is large enough to guarantee $|\nu^{[L]}_{\mathcal{T}_{L-1}}-\nu^{[L]}_{\infty}| < \nu^{[L]}_{\infty}/2$. Independence of $\varepsilon^{[L]}$ resp. $r_{\alpha,L}$ can be shown by an analogous reasoning (that also justifies $r_{\alpha,L} < \rho^{[L-1]}$).

To show that the second solution $\kappa^{[L]}_{\mathcal{T}_{L-1}}(z)$ does not exist in case (3) of Lemma 5.4, we argue as follows: We know by Lemma 5.3 that this is true for the limiting case where all $T_\ell \in \mathcal{T}_{L-1}$ are infinite, i.e., that $\kappa^{[L]}_{\infty}(z)$ lies outside the closed disk $\overline{\mathcal{D}}(0,\rho^{[L-1]})$ for $z\in\mathcal{D}(\alpha,\varepsilon)$ provided that $\alpha<\alpha_L$. However, since $\kappa^{[L]}_{\mathcal{T}_{L-1}}(\alpha)\ge\kappa^{[L]}_{\infty}(\alpha)$ for positive α, which is a straightforward consequence of $U^{[L]}_{\mathcal{T}_{L-1}}(x)\le U^{[L]}(x)$, see (5.26), it follows that $\kappa^{[L]}_{\mathcal{T}_{L-1}}(z)$ must also lie outside of $\overline{\mathcal{D}}(0,\rho^{[L-1]})$. Finally, the asserted radius of convergence of at least $\rho^{[L-1]}$ of $U^{[L]}_{\mathcal{T}_{L-1}}(s)$ follows immediately from (5.24), and the algebraic singularity of $U^{[L]}(s)$ has already been established in the proof of Lemma 5.3.

What remains to be done is the derivation of the expressions for $\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)$ and $\kappa^{[L]}_{\mathcal{T}_{L-1}}(z)$ stated in item (1) of Lemma 5.4. Now, solving the defining functional equation (5.21) is equivalent to determining the functional inverse(s) of

$$z=\frac{s}{P_L\Big(B^{[L-1]}(s)-V^{[L-1]}_{\mathcal{T}_{L-1}}(s)\Big)}=\frac{s}{P_L\Big(B^{[L-1]}(s)\Big)}-v^{[L-1]}_{\mathcal{T}_{L-1}}(s)$$

for s near s_0, where

$$v^{[L-1]}_{\mathcal{T}_{L-1}}(s)=-s\frac{P_L'\Big(B^{[L-1]}(s)\Big)}{P_L\Big(B^{[L-1]}(s)\Big)^2}V^{[L-1]}_{\mathcal{T}_{L-1}}(s)+O\Big({V^{[L-1]}_{s_0,\mathcal{T}_{L-1}}}^2\Big)$$

by virtue of the Taylor expansion at $B^{[L-1]}(s)$ of the (denominator) above; $O(V^{[L-1]}_{s_0,\mathcal{T}_{L-1}})$ is a uniform bound on $V^{[L-1]}_{\mathcal{T}_{L-1}}(s)$ and hence on $v^{[L-1]}_{\mathcal{T}_{L-1}}(s)$, cf. Lemma 5.2

Hence, Lemma 4.9 applies (with z and s exchanged) for $f(s)=s/P_L\Big(B^{[L-1]}(s)\Big)$. The latter is equivalent to the functional equation $zP_L\Big(B^{[L-1]}(s)\Big)-s=0$ of Lemma 5.3, which is solved by $\zeta^{[L]}_{\infty}(z)$ and $\kappa^{[L]}_{\infty}(z)$. Using

$$f^{[-1]'}(s)=\frac{1}{f'\Big(f^{[-1]}(s)\Big)}, \tag{5.30}$$

Lemma 4.9 provides

$$\zeta^{[L]}_{T_{L-1}}(z) = \zeta^{[L]}_{\infty}(z) - \zeta^{[L]\prime}_{\infty}(z)\zeta^{[L]}_{\infty}(z)\frac{P_L'\Big(B^{[L-1]}\big(\zeta^{[L]}_{\infty}(z)\big)\Big)}{P_L\Big(B^{[L-1]}\big(\zeta^{[L]}_{\infty}(z)\big)\Big)^2} V^{[L-1]}_{T_{L-1}}\big(\zeta^{[L]}_{\infty}(z)\big) + O\big(V^{[L-1]\,2}_{\zeta,T_{L-1}}\big),$$

from where (5.22) follows easily. The derivation of the analogous expression (5.23) for $\kappa^{[L]}_{T_{L-1}}(z)$ is literally the same. Note, however, that the latter is only valid in case (1). This eventually completes the proof of Lemma 5.4. □

Note that Lemma 5.4 "degenerates" to Lemma 5.3 (and hence to Lemma 5.1) for $L = 1$, remember our comments following Lemma 5.3. Actually, the results of Lemma 5.4 remain valid also for $L = 1$ if we use the (natural) convention $V^{[0]}(z) \equiv 0$.

Having determined the properties of the solutions $\zeta^{[L]}_{T_{L-1}}(z)$, $\kappa^{[L]}_{T_{L-1}}(z)$ of the functional equation $zP_L\big(B^{[L-1]}_{T_{L-1}}(s)\big) - s = 0$, which primarily determine the asymptotics of $B^{[L]}_{T_L}(z)$, our next step is to express $B^{[L]}_{T_L}(z)$ in terms of these solutions:

Lemma 5.5. *The improper PGF of $\mathcal{T}_L$-feasible busy periods $B^{[L]}_{T_L}(z)$ for $L \geq 1$ is analytic in $\mathcal{D}(0, \xi^{[L]}_{T_L})$ for $\xi^{[L]}_{T_L} \geq \rho^{[L]}_{T_{L-1}}$, $\xi^{[L]}_{T_L} - \rho^{[L]}_{T_{L-1}} = O(1/T_L^2)$ for $T_L \to \infty$, with a polar singularity at $z = \xi^{[L]}_{T_L}$ on its circle of convergence if T_L is finite.*

For $z \in \mathcal{D}(\alpha, \varepsilon^{[L]}_{T_{L-1}})$, $0 < \alpha < \rho^{[L]}_{T_{L-1}}$, $\varepsilon^{[L]}_{T_{L-1}} > 0$ chosen sufficiently small, and α_L being defined as in Lemma 5.4, we have the following expansions:

(1) *If $\alpha_L < \alpha < \rho^{[L]}_{T_{L-1}}$,*

$$\begin{aligned} B^{[L]}_{T_L}(z) &= B^{[L-1]}_{T_{L-1}}\big(\zeta^{[L]}_{T_{L-1}}(z)\big) \\ &+ \frac{B^{[L-1]}_{T_{L-1}}\big(\zeta^{[L]}_{T_{L-1}}(z)\big)\Big(1 - zU^{[L]\,\prime}_{T_{L-1}}\big(\zeta^{[L]}_{T_{L-1}}(z)\big)\Big)}{B^{[L-1]}_{T_{L-1}}\big(\kappa^{[L]}_{T_{L-1}}(z)\big)\Big(1 - zU^{[L]\,\prime}_{T_{L-1}}\big(\kappa^{[L]}_{T_{L-1}}(z)\big)\Big)} \\ &\quad \cdot \Big(B^{[L-1]}_{T_{L-1}}\big(\kappa^{[L]}_{T_{L-1}}(z)\big) - B^{[L-1]}_{T_{L-1}}\big(\zeta^{[L]}_{T_{L-1}}(z)\big)\Big) \cdot \left(\frac{\kappa^{[L]}_{T_{L-1}}(z)}{\zeta^{[L]}_{T_{L-1}}(z)}\right)^{-(T_L-1)} \\ &+ O\left(\left(\frac{r_{\alpha,L}}{\zeta^{[L]}_{T_{L-1}}(z)}\right)^{-(T_L-1)}\right) \end{aligned} \tag{5.31}$$

for $T_L \to \infty$; the remainder term denotes an analytic function and is uniform for $z \in \mathcal{D}(\alpha, \varepsilon)$. Moreover, $\varepsilon > 0$ sufficiently small and $r_{\alpha,L} < \rho^{[L-1]}$ are independent of $\mathcal{T}_L$ if $\mathcal{T}_{L-1}$ is sufficiently large.

(2) *If $0 < \alpha \leq \alpha_L$ and all $\mathcal{T}_{L-1}$ are finite, we have the same expansion as in case (1) above, however, with the exception that $r_{\alpha,L} = r_{\alpha,L,\mathcal{T}_{L-1}}$ and $\varepsilon = \varepsilon^{[L]}_{\mathcal{T}_{L-1}}$ may depend on $\mathcal{T}_{L-1}$.*

(3) *If $0 < \alpha < \alpha_L$ and $\mathcal{T}_{L-1}$ sufficiently large is arbitrary (including some T_ℓ being infinite), we have the expansion*

$$B^{[L]}_{T_L}(z) = B^{[L-1]}_{T_{L-1}}\big(\zeta^{[L]}_{T_{L-1}}(z)\big) + O_1\left(N(z)T_L^{-3/2}\left(\frac{\rho^{[L-1]}}{\zeta^{[L]}_{T_{L-1}}(z)}\right)^{-(T_L-1)}\big(1 + O(1/T_L)\big)\right) \tag{5.32}$$

for $T_L \to \infty$, where

$$N(z) = B^{[L-1]}_{T_{L-1}}\left(\zeta^{[L]}_{T_{L-1}}(z)\right)\left[1 - zU^{[L]}_{T_{L-1}}{}'\left(\zeta^{[L]}_{T_{L-1}}(z)\right)\right]\frac{d^{[L-1]}\beta^{[L-1]}\rho^{[L-1]}}{2\sqrt{\pi}\tau^{[L-1]}}$$
$$\cdot\left(\frac{-B^{[L-1]}_{T_{L-1}}\left(\zeta^{[L]}_{T_{L-1}}(z)\right)}{\tau^{[L-1]}\left(\rho^{[L-1]} - zP_L(\tau^{[L-1]})\right)} + \frac{B^{[L-1]}_{T_{L-1}}\left(\zeta^{[L]}_{T_{L-1}}(z)\right)}{\tau^{[L-1]}\left(\rho^{[L-1]} - zp_{0,L}\right)}\right.$$
$$\left. - \frac{zP_L'(\tau^{[L-1]})\tau^{[L-1]}}{\left(\rho^{[L-1]} - zP_L(\tau^{[L-1]})\right)^2} + \frac{zB^{[L-1]}_{T_{L-1}}\left(\zeta^{[L]}_{T_{L-1}}(z)\right)P_L'(\tau^{[L-1]})}{\left(\rho^{[L-1]} - zP_L(\tau^{[L-1]})\right)^2}\right). \tag{5.33}$$

The remainder term denotes an analytic function and has an implied constant $M \leq 1 + \epsilon_O$ for some small epsilon$_O \geq 0$ that is uniform for $z \in \mathcal{D}(\alpha, \varepsilon)$.

For $\alpha = \rho^{[L]}_{T_{L-1}}$ and finite T_L, we have the expansion

$$B^{[L]}_{T_L}(z) = B^{[L-1]}_{T_{L-1}}(\tau^{[L]}_{T_{L-1}}) \tag{5.34}$$
$$\cdot\frac{1 + \frac{2\delta^{[L]}_{T_{L-1}}}{\beta^{[L]}_{T_{L-1}}}v\cot\left(\frac{\beta^{[L]}_{T_{L-1}}(T_L-1)v}{\tau^{[L]}_{T_{L-1}}}\right) + O\left(v^2\cot\left(\frac{\beta^{[L]}_{T_{L-1}}(T_L-1)v}{\tau^{[L]}_{T_{L-1}}}\right)\right)}{1 + \left(\frac{\beta^{[L]}_{T_{L-1}}B^{[L-1]\prime}_{T_{L-1}}(\tau^{[L]}_{T_{L-1}})}{B^{[L-1]}_{T_{L-1}}(\tau^{[L]}_{T_{L-1}})} + \frac{2\delta^{[L]}_{T_{L-1}}}{\beta^{[L]}_{T_{L-1}}}\right)v\cot\left(\frac{\beta^{[L]}_{T_{L-1}}(T_L-1)v}{\tau^{[L]}_{T_{L-1}}}\right) + O\left(v^2\cot\left(\frac{\beta^{[L]}_{T_{L-1}}(T_L-1)v}{\tau^{[L]}_{T_{L-1}}}\right)\right)}$$

uniformly valid for all $v = \sqrt{z/\rho^{[L]}_{T_{L-1}} - 1} = O(1/T_L)$ as $T_L \to \infty$; $\beta^{[L]}_{T_{L-1}}$ and $\delta^{[L]}_{T_{L-1}}$ are the coefficients in the expansion of $\zeta^{[L]}_{T_{L-1}}(z)$ according to Lemma 4.1. The polar singularity on the circle of convergence at $z = \xi^{[L]}_{T_L} > \rho^{[L]}_{T_{L-1}}$ results from the vanishing denominator.

Proof. For case (1) and (2) of our lemma, Lemma 4.5 is applied to the numerator resp. the first term of the denominator of $B^{[L]}_{T_L}(z)$ in Theorem 3.1. For the numerator, we have $n = T_L - 2$, $W(s) \equiv 1$, and $\zeta^{[L]}_{T_{L-1}}(z)$, $\kappa^{[L]}_{T_{L-1}}(z)$ established by Lemma 5.4, so that Lemma 4.5 may be applied without problems. The first term of the denominator differs in $n = T_L - 1$ and $W(s) = s/B^{[L-1]}_{T_{L-1}}(s)$ only. Here we must assure that there are no zeroes of $B^{[L-1]}_{T_{L-1}}(s)$ except at $s = 0$ for $s \in \overline{\mathcal{D}}(0, r_\alpha)$, but this is guaranteed by Lemma 4.4. Thus, we may apply Lemma 4.5 without problems again.

The second term of the denominator in Theorem 3.1 provides a smaller order term, since it is analytic where $B^{[L-1]}_{T_{L-1}}(s)$ is; note that there is no pole from $s - zp_{0,L}$ due to the simultaneously vanishing numerator. Recalling (5.1), we find

$$\frac{s}{B^{[L-1]}(s)} = \frac{\rho_l^{[L-1]} + O(1 - s/\rho_l^{[L-1]})}{\frac{\rho_l^{[L-1]}}{\rho^{[L-1]}}\left[\tau^{[L-1]} - \beta^{[L-1]}\cdot(1 - z/\rho_l^{[L-1]})^{1/2} + O\left((1 - z/\rho_l^{[L-1]})\right)\right]}$$
$$= \frac{\rho^{[L-1]}}{\tau^{[L-1]}}\cdot\frac{1 + O(1 - s/\rho_l^{[L-1]})}{1 - \beta^{[L-1]}/\tau^{[L-1]}\cdot(1 - z/\rho_l^{[L-1]})^{1/2} + O\left((1 - z/\rho_l^{[L-1]})\right)}$$
$$= \frac{\rho^{[L-1]}}{\tau^{[L-1]}} + \frac{\rho^{[L-1]}\beta^{[L-1]}}{(\tau^{[L-1]})^2}(1 - z/\rho_l^{[L-1]})^{1/2} + O\left((1 - z/\rho_l^{[L-1]})\right) \tag{5.35}$$

for $z \to \rho_l^{[L-1]}$. If $B_{T_{L-1}}^{[L-1]}(s)$ is analytic at the singular points $z = \rho_l^{[L-1]}$ of $B^{[L-1]}(s)$, the same argument as used in the proof of Corollary 4.8 shows that

$$\frac{s}{B_{T_{L-1}}^{[L-1]}(s)} - \frac{\rho_l^{[L-1]}}{B_{T_{L-1}}^{[L-1]}(\rho_l^{[L-1]})} = o\left(\frac{s}{B^{[L-1]}(s)} - \frac{\rho^{[L-1]}}{\tau^{[L-1]}}\right) \qquad \text{for } s \to \rho_l^{[L-1]} \text{ in } \Delta_{\rho^{[L-1]}}.$$

In any case, we obtain

$$[s^{T_L-1}]\frac{s}{B_{T_{L-1}}^{[L-1]}(s)} = O_1\left(\frac{-d^{[L-1]}\rho^{[L-1]}\beta^{[L-1]}}{2\sqrt{\pi}(\tau^{[L-1]})^2}T_L^{-3/2}(\rho^{[L-1]})^{-(T_L-1)}\Big(1+O(1/T_L)\Big)\right), \quad (5.36)$$

where $d^{[L-1]} = \gcd(P^{[L-1]})$, cf. Lemma 5.1. Since $|\kappa_\infty^{[L]}(z)| < r_{\alpha,L} < \rho^{[L-1]}$ by Lemma 5.3, it thus follows that the contribution (5.36) of the second term of the denominator of $B_{T_L}^{[L]}(z)$ in Theorem 3.1 vanishes in the remainder term provided by the former applications of Lemma 4.5.

Putting everything together, the expansion of $B_{T_L}^{[L]}(z)$ for $T_L \to \infty$ in case (1) and (2) of Lemma 5.5 reads

$$B_{T_L}^{[L]}(z) = B_{T_{L-1}}^{[L-1]}\Big(\zeta_{T_{L-1}}^{[L]}(z)\Big)$$
$$\cdot \frac{\dfrac{\zeta_{T_{L-1}}^{[L]}(z)^{-T_L+1}}{1-zU_{T_{L-1}}^{[L]}{}'(\zeta_{T_{L-1}}^{[L]}(z))} + \dfrac{\kappa_{T_{L-1}}^{[L]}(z)^{-T_L+1}}{1-zU_{T_{L-1}}^{[L]}{}'(\kappa_{T_{L-1}}^{[L]}(z))} + O({r_{\alpha,L}}^{-T_L+2})}{\dfrac{\zeta_{T_{L-1}}^{[L]}(z)^{-T_L+1}}{1-zU_{T_{L-1}}^{[L]}{}'(\zeta_{T_{L-1}}^{[L]}(z))} + \dfrac{B_{T_{L-1}}^{[L-1]}(\zeta_{T_{L-1}}^{[L]}(z))}{B_{T_{L-1}}^{[L-1]}(\kappa_{T_{L-1}}^{[L]}(z))}\cdot\dfrac{\kappa_{T_{L-1}}^{[L]}(z)^{-T_L+1}}{1-zU_{T_{L-1}}^{[L]}{}'(\kappa_{T_{L-1}}^{[L]}(z))} + O({r_{\alpha,L}}^{-T_L+1})}.$$

In case of $\alpha < \rho_{T_{L-1}}^{[L]}$, this may be further evaluated since we know that $\zeta_{T_{L-1}}^{[L]}(x) < \kappa_{T_{L-1}}^{[L]}(x)$ for real positive x. We thus obtain

$$B_{T_L}^{[L]}(z) = B_{T_{L-1}}^{[L-1]}\Big(\zeta_{T_{L-1}}^{[L]}(z)\Big)$$
$$\cdot \frac{1 + \dfrac{\kappa_{T_{L-1}}^{[L]}(z)^{-T_L+1}\Big(1-zU_{T_{L-1}}^{[L]}{}'(\zeta_{T_{L-1}}^{[L]}(z))\Big)}{\zeta_{T_{L-1}}^{[L]}(z)^{-T_L+1}\Big(1-zU_{T_{L-1}}^{[L]}{}'(\kappa_{T_{L-1}}^{[L]}(z))\Big)} + O\left(\left(\dfrac{r_{\alpha,L}}{\zeta_{T_{L-1}}^{[L]}(z)}\right)^{-(T_L-1)}\right)}{1 + \dfrac{B_{T_{L-1}}^{[L-1]}(\zeta_{T_{L-1}}^{[L]}(z))\kappa_{T_{L-1}}^{[L]}(z)^{-T_L+1}\Big(1-zU_{T_{L-1}}^{[L]}{}'(\zeta_{T_{L-1}}^{[L]}(z))\Big)}{B_{T_{L-1}}^{[L-1]}(\kappa_{T_{L-1}}^{[L]}(z))\zeta_{T_{L-1}}^{[L]}(z)^{-T_L+1}\Big(1-zU_{T_{L-1}}^{[L]}{}'(\kappa_{T_{L-1}}^{[L]}(z))\Big)} + O\left(\left(\dfrac{r_{\alpha,L}}{\zeta_{T_{L-1}}^{[L]}(z)}\right)^{-(T_L-1)}\right)}$$

$$= B_{T_{L-1}}^{[L-1]}\Big(\zeta_{T_{L-1}}^{[L]}(z)\Big)$$
$$\cdot\left(1 + \frac{\kappa_{T_{L-1}}^{[L]}(z)^{-(T_L-1)}\Big(1 - zU_{T_{L-1}}^{[L]}{}'\big(\zeta_{T_{L-1}}^{[L]}(z)\big)\Big)}{\zeta_{T_{L-1}}^{[L]}(z)^{-(T_L-1)}\Big(1 - zU_{T_{L-1}}^{[L]}{}'\big(\kappa_{T_{L-1}}^{[L]}(z)\big)\Big)}\right.$$
$$- \frac{B_{T_{L-1}}^{[L-1]}\big(\zeta_{T_{L-1}}^{[L]}(z)\big)\kappa_{T_{L-1}}^{[L]}(z)^{-(T_L-1)}\Big(1 - zU_{T_{L-1}}^{[L]}{}'\big(\zeta_{T_{L-1}}^{[L]}(z)\big)\Big)}{B_{T_{L-1}}^{[L-1]}\big(\kappa_{T_{L-1}}^{[L]}(z)\big)\zeta_{T_{L-1}}^{[L]}(z)^{-(T_L-1)}\Big(1 - zU_{T_{L-1}}^{[L]}{}'\big(\kappa_{T_{L-1}}^{[L]}(z)\big)\Big)}$$
$$\left. + O\left(\left(\frac{r_{\alpha,L}}{\zeta_{T_{L-1}}^{[L]}(z)}\right)^{-(T_L-1)}\right)\right);$$

some algebraic manipulations eventually establish the expansion (5.31).

In case (3) of our lemma, only $\zeta^{[L]}_{T_{L-1}}(z)$ exists for $s \in \mathcal{D}(0, \rho^{[L-1]})$ by Lemma 5.4. In order to apply Corollary 4.8 (resp. Lemma 4.7), we have to provide expansions of the functions involved in the numerator resp. denominator of $B^{[L]}_{T_L}(z)$ in Theorem 3.1 for $s \to \rho_l^{[L-1]}$. For the numerator, we have $n = T_L - 2$, $W(s) \equiv 1$, and $U(s) = P_L\big(B^{[L-1]}_{T_{L-1}}(s)\big)$. However, according to Corollary 4.8, we may rely upon $U(s) = P_L\big(B^{[L-1]}(s)\big)$ instead. Hence, w.r.t. (4.16), we find $w(\rho_l^{[L-1]}) = 1$, $x(\rho_l^{[L-1]}) = 0$, and

$$u(\rho_l^{[L-1]}) = P_L\Big(\frac{\rho_l^{[L-1]}}{\rho^{[L-1]}}\tau^{[L-1]}\Big) \qquad \text{and} \qquad v(\rho_l^{[L-1]}) = P_L'\Big(\frac{\rho_l^{[L-1]}}{\rho^{[L-1]}}\tau^{[L-1]}\Big)\frac{\rho_l^{[L-1]}\beta^{[L-1]}}{\rho^{[L-1]}}$$

by virtue of

$$P_L\big(B^{[L-1]}(s)\big) = P_L\Big(\frac{\rho_l^{[L-1]}}{\rho^{[L-1]}}\tau^{[L-1]}\Big) - P_L'\Big(\frac{\rho_l^{[L-1]}}{\rho^{[L-1]}}\tau^{[L-1]}\Big)\frac{\rho_l^{[L-1]}\beta^{[L-1]}}{\rho^{[L-1]}}(1 - s/\rho_l^{[L-1]})^{1/2}$$
$$+ O\big((1 - s/\rho_l^{[L-1]})\big) \qquad \text{for } s \to \rho_l^{[L-1]}, \tag{5.38}$$

where we used (5.1) and the Taylor expansion of $P_L(z)$ at $\frac{\rho_l^{[L-1]}}{\rho^{[L-1]}}\tau^{[L-1]}$. Plugging this into (4.19) yields the contribution $R_N(z) = O_1\Big(M_N(z)T_L^{-3/2}(\rho^{[L-1]})^{-(T_L-1)}\big(1 + O(1/T_L)\big)\Big)$ with

$$M_N(z) = \frac{-d^{[L-1]}}{2\sqrt{\pi}} \cdot \frac{zP_L'(\tau^{[L-1]})\beta^{[L-1]}\rho^{[L-1]}}{\big(\rho^{[L-1]} - zP_L(\tau^{[L-1]})\big)^2} \tag{5.39}$$

for $T_L \to \infty$.

The first term of the denominator in Theorem 3.1 differs from the numerator considered before solely in $n = T_L - 1$ and $W(s) = s/B^{[L-1]}_{T_{L-1}}(s)$. Expansion (5.35) immediately provides $w(\rho_l^{[L-1]}) = \rho^{[L-1]}/\tau^{[L-1]}$ and $x(\rho_l^{[L-1]}) = -\rho^{[L-1]}\beta^{[L-1]}/(\tau^{[L-1]})^2$. This yields the contribution $R_D(z) = O_1\Big(M_D(z)T_L^{-3/2}(\rho^{[L-1]})^{-(T_L-1)}\big(1 + O(1/T_L)\big)\Big)$ with

$$M_D(z) = \frac{\frac{d^{[L-1]}\rho^{[L-1]}}{\tau^{[L-1]}}}{\rho^{[L-1]} - zP_L(\tau^{[L-1]})} \cdot \frac{-1}{2\sqrt{\pi}}\Bigg[-\frac{\beta^{[L-1]}}{\tau^{[L-1]}} + \frac{zP_L'(\tau^{[L-1]})\beta^{[L-1]}}{\rho^{[L-1]} - zP_L(\tau^{[L-1]})}\Bigg]$$
$$= \frac{-d^{[L-1]}\beta^{[L-1]}\rho^{[L-1]}}{2\sqrt{\pi}\tau^{[L-1]}\big(\rho^{[L-1]} - zP_L(\tau^{[L-1]})\big)} \cdot \Bigg[\frac{-1}{\tau^{[L-1]}} + \frac{zP_L'(\tau^{[L-1]})}{\rho^{[L-1]} - zP_L(\tau^{[L-1]})}\Bigg]$$

for $T_L \to \infty$.

Finally, the contribution arising from the second term of the denominator has almost been established in (5.36): Since $\alpha < \rho^{[L]}_{T_{L-1}}$ allows us to assume $|z| < \rho^{[L]} < \rho^{[L-1]}$ for T_{L-1} sufficiently large, see Lemma 5.4 resp. Lemma 5.1, we have $zp_{0,L} < \rho^{[L-1]}$. Hence,

$$\frac{s/B^{[L-1]}_{T_{L-1}}(s) - zp_{0,L}/B^{[L-1]}_{T_{L-1}}(zp_{0,L})}{s - zp_{0,L}} = \frac{1}{\rho_l^{[L-1]} - zp_{0,L}} \cdot \frac{s}{B^{[L-1]}_{T_{L-1}}(s)} + O(1)$$

for $s \to \rho_l^{[L-1]}$, so that expansion (5.36) provided earlier reveals its contribution as $R_H(z) = O_1\Big(M_H(z)T_L^{-3/2}(\rho^{[L-1]})^{-(T_L-1)}\big(1+O(1/T_L)\big)\Big)$ with

$$M_H(z) = \frac{-d^{[L-1]}\rho^{[L-1]}\beta^{[L-1]}}{2\sqrt{\pi}(\tau^{[L-1]})^2(\rho^{[L-1]} - zp_{0,L})}$$

for $T_L \to \infty$.

Putting everything together, Lemma 4.7 (resp. Corollary 4.8) yields

$$\begin{aligned} B_{T_L}^{[L]}(z) &= \frac{\frac{\zeta_{T_{L-1}}^{[L]}(z)^{-T_L+1}}{1-zU_{T_{L-1}}^{[L]}{}'(\zeta_{T_{L-1}}^{[L]}(z))} + R_N(z)}{\frac{1}{B_{T_{L-1}}^{[L-1]}(\zeta_{T_{L-1}}^{[L]}(z))}\cdot\frac{\zeta_{T_{L-1}}^{[L]}(z)^{-T_L+1}}{1-zU_{T_{L-1}}^{[L]}{}'(\zeta_{T_{L-1}}^{[L]}(z))} + R_D(z) + R_H(z)} \\ &= B_{T_{L-1}}^{[L-1]}\big(\zeta_{T_{L-1}}^{[L]}(z)\big) + O_1\left(N(z)T_L^{-3/2}\big(1+O(1/T_L)\big)\left(\frac{\rho^{[L-1]}}{\zeta_{T_{L-1}}^{[L]}(z)}\right)^{-(T_L-1)}\right) \end{aligned}$$

with

$$\begin{aligned} N(z) = {} & B_{T_{L-1}}^{[L-1]}\big(\zeta_{T_{L-1}}^{[L]}(z)\big)\Big[1 - zU_{T_{L-1}}^{[L]}{}'\big(\zeta_{T_{L-1}}^{[L]}(z)\big)\Big] \\ & \cdot\Big(M_N(z) - B_{T_{L-1}}^{[L-1]}\big(\zeta_{T_{L-1}}^{[L]}(z)\big)\big(M_D(z) + M_H(z)\big)\Big) \end{aligned}$$

Noting

$$\begin{aligned} M_D(z) + M_H(z) = {} & \frac{-d^{[L-1]}\beta^{[L-1]}\rho^{[L-1]}}{2\sqrt{\pi}\tau^{[L-1]}}\cdot\left(\frac{-1}{\tau^{[L-1]}\big(\rho^{[L-1]} - zP_L(\tau^{[L-1]})\big)}\right. \\ & \left. + \frac{1}{\tau^{[L-1]}\big(\rho^{[L-1]} - zp_{0,L}\big)} + \frac{zP_L'(\tau^{[L-1]})}{\big(\rho^{[L-1]} - zP_L(\tau^{[L-1]})\big)^2}\right) \end{aligned}$$

and recalling (5.39), our result (5.32) follows by some straightforward algebraic manipulations.

Finally, for α in a neighborhood of $\rho_{T_{L-1}}^{[L]}$, Lemma 4.6 provides the required framework. However, T_L must be finite to guarantee a non-zero region of validity! For convenience, we think of applying Lemma 4.5 to the result of Theorem 3.1 again, this time looking at the coefficient of $[s^{T_L-2}]$ in the nominator and denominator, respectively, but therefore omitting the single s in the denominator's $W(s)$: For the numerator, we have $W(s) \equiv 1$ and hence $W'(s)/W(s) \equiv 0$, for the denominator, we have $W(s) = 1/B_{T_{L-1}}^{[L-1]}(s)$ and hence $W'(s)/W(s) = -B_{T_{L-1}}^{[L-1]}{}'(s)/B_{T_{L-1}}^{[L-1]}(s)$. Applying Lemma 4.6, the expansion stated in Lemma 5.5 follows immediately; note that the contribution of the second term in the denominator of Theorem 3.1 is absorbed since $\rho^{[L-1]} > r_{\alpha,L} > \tau_{T_{L-1}}^{[L]}$.

What remains to be done is to show that $B_{T_L}^{[L]}(z)$ is analytic in $\mathcal{D}(0,\xi_{T_L}^{[L]})$. First of all, we know that $B_{T_L}^{[L]}(z)$ is analytic in $\mathcal{D}(\alpha,\varepsilon)$ for any $0 < \alpha < \rho_{T_{L-1}}^{[L]}$ (and of course also for $\alpha = 0$) and has non-negative Taylor coefficients, so it must have a singularity at $x \geq \rho_{T_{L-1}}^{[L]}$ by Pringsheim's theorem. From expansion (5.34) it is immediately apparent that there is indeed a (simple) pole from the denominator vanishing at some $\xi_{T_L}^{[L]} > \rho_{T_{L-1}}^{[L]}$ for finite[7] T_L: If we increase v from $\rho_{T_{L-1}}^{[L]}$ onwards, the argument of the cotangens approaches π approximately at $v = O(1/T_L)$, yielding a cancellation of the terms in the denominator; note that it is not possible for the numerator to vanish simultaneously.

This eventually completes the proof of Lemma 5.5. □

Note that Lemma 5.5 covers the case $L = 1$ as well, cf. our remarks following Lemma 5.4 and Lemma 5.3. Again, we need the convention $B^{[0]}(z) = z$ according to Theorem 3.1 for this purpose.

We now combine the results of Lemma 5.5 and Lemma 5.4 for case (1), where both solutions $\zeta_{T_{L-1}}^{[L]}(z)$ and $\kappa_{T_{L-1}}^{[L]}(z)$ exist:

Lemma 5.6. *Let α_L be defined as in Lemma 5.4 and $z \in \mathcal{D}(\alpha, \varepsilon)$, $\alpha_L < \alpha < \rho^{[L]}$. Abbreviating*

$$K_L(s) = K^{[L]}\Big(\frac{s}{P^{[L]}(s)}\Big),$$

such that $K_L\big(B^{[L]}(z)\big) = K^{[L]}(z)$, we have for $L \geq 1$

$$\begin{aligned} V_{T_L}^{[L]}(z) = {} & \frac{1 - \frac{B^{[L]}(z)}{P^{[L-1]}(B^{[L]}(z))} P^{[L-1]\prime}\big(B^{[L]}(z)\big)}{1 - \frac{B^{[L]}(z)}{P^{[L]}(B^{[L]}(z))} P^{[L]\prime}\big(B^{[L]}(z)\big)} V_{T_{L-1}}^{[L-1]}\Big(\frac{B^{[L]}(z)}{P^{[L-1]}\big(B^{[L]}(z)\big)}\Big) \\ & - \frac{1 - \frac{B^{[L]}(z)}{P^{[L]}(B^{[L]}(z))} P^{[L]\prime}\big(B^{[L]}(z)\big)}{1 - \frac{B^{[L]}(z)}{P^{[L-1]}(B^{[L]}(z))} P^{[L-1]\prime}\big(B^{[L]}(z)\big)} \\ & \cdot \frac{1 - \frac{B^{[L]}(z)}{P^{[L]}(B^{[L]}(z))} P_L\Big(K_L\big(B^{[L]}(z)\big)\Big) P^{[L-1]\prime}\Big(K_L\big(B^{[L]}(z)\big)\Big)}{1 - \frac{B^{[L]}(z)}{P^{[L]}(B^{[L]}(z))} P^{[L]\prime}\Big(K_L\big(B^{[L]}(z)\big)\Big)} \\ & \cdot \Big(B^{[L]}(z) - \frac{B^{[L]}(z)^2}{K_L\big(B^{[L]}(z)\big)}\Big)\Big(\frac{P_L\Big(K_L\big(B^{[L]}(z)\big)\Big)}{P_L\big(B^{[L]}(z)\big)}\Big)^{-(T_L-1)}\Big(1 + O\big(T_L V_{\kappa,T_{L-1}}^{[L-1]}\big)\Big) \\ & + O\big(V_{\zeta,T_{L-1}}^{[L-1]\,2}\big) + O\bigg(\Big(\frac{r_{\alpha,L}}{zP_L\big(B^{[L]}(z)\big)}\Big)^{-(T_L-1)}\bigg) \end{aligned}$$

for $T_L \to \infty$ in a way that ensures $T_L V_{\kappa,T_{L-1}}^{[L-1]} = o(1)$, where $V_{\kappa,T_{L-1}}^{[L-1]}$, $V_{\zeta,T_{L-1}}^{[L-1]}$ have been defined in Lemma 5.4, $V^{[0]}(z) \equiv 0$, and $P^{[0]}(z) \equiv 1$.

---

[7]Actually, (the derivation of) expansion (5.34) answers two mathematical problems that puzzled us quite a long time: (1) How is it possible for $B_{T_L}^{[L]}(z)$ to have a radius of convergence $\xi_{T_L}^{[L]} > \rho_{T_{L-1}}^{[L]}$, although the solutions $\zeta_{T_{L-1}}^{[L]}(z)$ and $\kappa_{T_{L-1}}^{[L]}(z)$ involved in the expression have a radius of convergence of $\rho_{T_{L-1}}^{[L]}$, and (2) how does $B_{T_L}^{[L]}(z)$, which is rational for finite T_L and hence takes arbitrary large values, actually approach its limit $B^{[L]}(z)$, which has an algebraic singularity at $z = \rho^{[L]}$ and is hence incapable of taking values larger than $\tau^{[L]} = B^{[L]}(\rho^{[L]})$?

Proof. Recalling the expression (5.22) for $\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)$ and the defining functional equation (5.21), we obtain by using (5.30) and (5.6)

$$\begin{aligned}
B^{[L-1]}_{\mathcal{T}_{L-1}}\Big(\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)\Big) &= P^{[-1]}_L\Big(\frac{\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)}{z}\Big) \\
&= P^{[-1]}_L\Big(\frac{\zeta^{[L]}_\infty(z)}{z}\Big) - P^{[-1]\prime}_L\Big(\frac{\zeta^{[L]}_\infty(z)}{z}\Big)\frac{z\zeta^{[L]\prime}_\infty(z)}{\zeta^{[L]}_\infty(z)}P'_L\Big(B^{[L-1]}\big(\zeta^{[L]}_\infty(z)\big)\Big) \\
&\qquad \cdot V^{[L-1]}_{\mathcal{T}_{L-1}}\Big(\zeta^{[L]}_\infty(z)\Big) + O\Big(V^{[L-1]\,2}_{\zeta,\mathcal{T}_{L-1}}\Big) \\
&= B^{[L-1]}\Big(\zeta^{[L]}_\infty(z)\Big) - \frac{z\zeta^{[L]\prime}_\infty(z)}{\zeta^{[L]}_\infty(z)}V^{[L-1]}_{\mathcal{T}_{L-1}}\Big(\zeta^{[L]}_\infty(z)\Big) + O\Big(V^{[L-1]\,2}_{\zeta,\mathcal{T}_{L-1}}\Big)
\end{aligned}$$

uniformly for $\mathcal{T}_{L-1}\to\infty$. Remembering (4.15), we also have

$$\frac{1}{1-zU^{[L]\ \prime}_{\mathcal{T}_{L-1}}\Big(\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)\Big)} = \frac{z\zeta^{[L]\ \prime}_{\mathcal{T}_{L-1}}(z)}{\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)} = \frac{z\zeta^{[L]\prime}_\infty(z)}{\zeta^{[L]}_\infty(z)} + O\Big(V^{[L-1]}_{\zeta,\mathcal{T}_{L-1}}\Big)$$

for $\mathcal{T}_{L-1}\to\infty$. Clearly, analogous expressions hold for $\kappa^{[L]}_{\mathcal{T}_{L-1}}(z)$.

Since $O(V^{[L-1]}_{\kappa,\mathcal{T}_{L-1}})$ dominates $O(V^{[L-1]}_{\zeta,\mathcal{T}_{L-1}})$, cf. item (1) of Lemma 5.4, we find

$$\left(\frac{\kappa^{[L]}_{\mathcal{T}_{L-1}}(z)}{\zeta^{[L]}_{\mathcal{T}_{L-1}}(z)}\right)^{-n} = \left(\frac{\kappa^{[L]}_\infty(z)+O\Big(V^{[L-1]}_{\kappa,\mathcal{T}_{L-1}}\Big)}{\zeta^{[L]}_\infty(z)+O\Big(V^{[L-1]}_{\zeta,\mathcal{T}_{L-1}}\Big)}\right)^{-n} = \left(\frac{\kappa^{[L]}_\infty(z)}{\zeta^{[L]}_\infty(z)}\right)^{-n}\Big(1+O\Big(nV^{[L-1]}_{\kappa,\mathcal{T}_{L-1}}\Big)\Big)$$

for (reasonable) values of n such that $nV^{[L-1]}_{\kappa,\mathcal{T}_{L-1}} = o(1)$ as $\mathcal{T}_{L-1}\to\infty$ and $n\to\infty$.

Plugging these results into expression (5.31) yields

$$\begin{aligned}
B^{[L]}_{\mathcal{T}_L}(z) &= B^{[L-1]}\Big(\zeta^{[L]}_\infty(z)\Big) - \frac{z\zeta^{[L]\prime}_\infty(z)}{\zeta^{[L]}_\infty(z)}V^{[L-1]}_{\mathcal{T}_{L-1}}\Big(\zeta^{[L]}_\infty(z)\Big) + O\Big(V^{[L-1]\,2}_{\zeta,\mathcal{T}_{L-1}}\Big) \\
&+ \frac{B^{[L-1]}\Big(\zeta^{[L]}_\infty(z)\Big)\frac{z\kappa^{[L]\prime}_\infty(z)}{\kappa^{[L]}_\infty(z)}}{B^{[L-1]}\Big(\kappa^{[L]}_\infty(z)\Big)\frac{z\zeta^{[L]\prime}_\infty(z)}{\zeta^{[L]}_\infty(z)}}\Big(B^{[L-1]}\Big(\kappa^{[L]}_\infty(z)\Big) - B^{[L-1]}\Big(\zeta^{[L]}_\infty(z)\Big)\Big) \\
&\cdot\left(\frac{\kappa^{[L]}_\infty(z)}{\zeta^{[L]}_\infty(z)}\right)^{-(T_L-1)}\Big(1+O\Big(T_LV^{[L-1]}_{\kappa,\mathcal{T}_{L-1}}\Big)\Big) + O\left(\left(\frac{r_{\alpha,L}}{zP_L\Big(B^{[L]}(z)\Big)}\right)^{-(T_L-1)}\right).
\end{aligned} \tag{5.40}$$

Using the properties (5.9)–(5.12) and, according to (5.5),

$$B^{[L-1]\prime}\Big(\zeta^{[L]}_\infty(z)\Big) = \frac{P^{[L-1]}\Big(B^{[L-1]}\Big(\zeta^{[L]}_\infty(z)\Big)\Big)}{1-\zeta^{[L]}_\infty(z)P^{[L-1]\prime}\Big(B^{[L-1]}\Big(\zeta^{[L]}_\infty(z)\Big)\Big)} = \frac{P^{[L-1]}\Big(B^{[L]}(z)\Big)}{1-\zeta^{[L]}_\infty(z)P^{[L-1]\prime}\Big(B^{[L]}(z)\Big)}$$

we find by (4.15)

$$
\begin{aligned}
\frac{z\zeta_\infty^{[L]\prime}(z)}{\zeta_\infty^{[L]}(z)} &= \frac{1}{1 - zU^{[L]\prime}\big(\zeta_\infty^{[L]}(z)\big)} \qquad (5.41)\\
&= \frac{1}{1 - zP_L'\Big(B^{[L-1]}\big(\zeta_\infty^{[L]}(z)\big)\Big)B^{[L-1]\prime}\big(\zeta_\infty^{[L]}(z)\big)}\\
&= \frac{1 - \zeta_\infty^{[L]}(z)P^{[L-1]\prime}\big(B^{[L]}(z)\big)}{1 - \zeta_\infty^{[L]}(z)P^{[L-1]\prime}\big(B^{[L]}(z)\big) - zP_L'\big(B^{[L]}(z)\big)P^{[L-1]}\big(B^{[L]}(z)\big)}\\
&= \frac{1 - zP_L\big(B^{[L]}(z)\big)P^{[L-1]\prime}\big(B^{[L]}(z)\big)}{1 - zP^{[L]\prime}\big(B^{[L]}(z)\big)}.
\end{aligned}
$$

An analogous derivation provides

$$
\frac{z\kappa_\infty^{[L]\prime}(z)}{\kappa_\infty^{[L]}(z)} = \frac{1}{1 - zU^{[L]\prime}\big(\kappa_\infty^{[L]}(z)\big)} = \frac{1 - zP_L\big(K^{[L]}(z)\big)P^{[L-1]\prime}\big(K^{[L]}(z)\big)}{1 - zP^{[L]\prime}\big(K^{[L]}(z)\big)}. \qquad (5.42)
$$

Plugging the above results into (5.40) and using the simple fact

$$
z = \frac{B^{[L]}(z)}{P^{[L]}\big(B^{[L]}(z)\big)}, \qquad (5.43)
$$

cf. Lemma 5.1, some algebraic manipulations yield the expression given in Lemma 5.6. □

Note that we introduced the function $K_L\big(B^{[L]}(z)\big) = K^{[L]}(z)$ to achieve that z appears only in conjunction with $B^{[L]}(z)$, a property that greatly simplifies our subsequent derivations.

The following lemma is analogous to Lemma 5.6 for case (3) of Lemma 5.5, where only $\zeta_{T_{L-1}}^{[L]}(z)$ exists. Recall that we are aiming at a reasonably accurate remainder term here.

Lemma 5.7. *Let α_L be defined as in Lemma 5.4. For $z \in \mathcal{D}(\alpha, \varepsilon)$ and $0 < \alpha < \alpha_L$, we have for $L \geq 1$*

$$
\begin{aligned}
V_{T_L}^{[L]}(z) &= \frac{1 - \frac{B^{[L]}(z)}{P^{[L-1]}(B^{[L]}(z))}P^{[L-1]\prime}\big(B^{[L]}(z)\big)}{1 - \frac{B^{[L]}(z)}{P^{[L]}(B^{[L]}(z))}P^{[L]\prime}\big(B^{[L]}(z)\big)} V_{T_{L-1}}^{[L-1]}\left(\frac{B^{[L]}(z)}{P^{[L-1]}\big(B^{[L]}(z)\big)}\right) \qquad (5.44)\\
&\quad + O\Big(V_{\zeta,T_{L-1}}^{[L-1]\,2}\Big) + O_1\left(M_L\left(\frac{B^{[L]}(z)}{P^{[L]}\big(B^{[L]}(z)\big)}\right)T_L^{-3/2}\left(\frac{\rho^{[L-1]}}{zP_L\big(B^{[L]}(z)\big)}\right)^{-(T_L-1)}\right)
\end{aligned}
$$

for $\mathcal{T}_L \to \infty$, *where* $V^{[0]}(z) \equiv 0$, $P^{[0]}(z) \equiv 1$, *and*

$$M_L(z) = \frac{B^{[L]}(z)\left[1 - zP^{[L]\prime}\left(B^{[L]}(z)\right)\right]}{1 - zP_L\left(B^{[L]}(z)\right)P^{[L-1]\prime}\left(B^{[L]}(z)\right)} \cdot d^{[L-1]}\sqrt{\frac{1}{2\pi P^{[L-1]}(\tau^{[L-1]})P^{[L-1]\prime\prime}(\tau^{[L-1]})}}$$
$$\cdot\left(\frac{-B^{[L]}(z)}{\tau^{[L-1]}\left(\rho^{[L-1]} - zP_L(\tau^{[L-1]})\right)} + \frac{B^{[L]}(z)}{\tau^{[L-1]}\left(\rho^{[L-1]} - zp_{0,L}\right)}\right.$$
$$\left. - \frac{zP_L'(\tau^{[L-1]})\left(\tau^{[L-1]} - B^{[L]}(z)\right)}{\left(\rho^{[L-1]} - zP_L(\tau^{[L-1]})\right)^2}\right) \qquad \text{for } L \geq 2, \tag{5.45}$$
$$M_1(z) \equiv 0.$$

Proof. According to Lemma 5.5, the major term in the expansion of $B^{[L]}_{T_L}(z)$ in case (3) is the same as in (5.31) for case (1), except that the term involving $\kappa^{[L]}_{T_{L-1}}(z)$ is missing and that there is a different remainder. Hence, the derivations in the proof of Lemma 5.6 translate literally.

The expression for $M_L(z)$ follows from (5.33) by replacing $1 - zU^{[L]}_{T_{L-1}}{}'\left(\zeta^{[L]}_{T_{L-1}}(z)\right)$ resp. $B^{[L-1]}_{T_{L-1}}\left(\zeta^{[L]}_{T_{L-1}}(z)\right)$ by $1 - zU^{[L]\prime}\left(\zeta^{[L]}_{\infty}(z)\right)$ resp. $B^{[L-1]}\left(\zeta^{[L]}_{\infty}(z)\right) = B^{[L]}(z)$, which has an neglible (exponentially small) effect $O(V^{[L-1]}_{\zeta,T_{L-1}})$ on the implied constant in O_1. Using (5.41) and

$$\frac{d^{[L-1]}\beta^{[L-1]}\rho^{[L-1]}}{2\sqrt{\pi}\tau^{[L-1]}} = d^{[L-1]}\sqrt{\frac{1}{2\pi P^{[L-1]}(\tau^{[L-1]})P^{[L-1]\prime\prime}(\tau^{[L-1]})}}$$

resulting from (4.4) for $U(s) = P^{[L-1]}(s)$, (5.45) follows immediately..

In case (2) of Lemma 5.5, where all $T_1, \ldots, T_L$ are finite, it is possible to give a slightly better characterization of the remainder involved. In this case, we know that $\kappa^{[L]}_{T_{L-1}}(z)$ exists, of course determining the actual remainder. From the proof of case (3) of Lemma 5.4, we know that $\kappa^{[L]}_{T_{L-1}}(z)$ lies outside of the disk $\mathcal{D}(0, \rho^{[L-1]})$ for $z \in \mathcal{D}(\alpha, \varepsilon)$, $\alpha < \alpha_L$. On the other hand, we know that $\kappa^{[L]}_{T_{L-1}}(z)$ must lie within the disk $\mathcal{D}(0, \xi^{[L-1]}_{T_{L-1}})$, since $\xi^{[L-1]}_{T_{L-1}} = \rho^{(L-1)}_{T_{L-2}} + O(T^{-2}_{L-1}) = \rho^{[L-1]} + O\left(V^{(L-2)}_{T_{L-2}}\right) + O(T^{-2}_{L-1})$ for $T_1, \ldots, T_{L-2}$ sufficiently large is the radius of convergence of $B^{[L-1]}_{T_{L-1}}(z)$, by expansion (5.34) applied for Level $L-1$. Hence it follows immediately that

$$\kappa^{[L]}_{T_{L-1}}(z) = \rho^{[L-1]}\left(1 + O\left(V^{(L-2)}_{T_{L-2}}\right) + O(T^{-2}_{L-1})\right).$$

A more refined treatment of (5.34) would of course provide more accurate information. Anyway, the conservative remainder term established by Corollary 4.8 is sufficient for our purposes. □

Lemmas 4.15 and 4.16 provide recursive formulas for $V^{[L]}_{T_L}(z)$ and hence $B^{[L]}_{T_L}(z)$, which are solved in the following lemma:

Lemma 5.8. *Let* $\mathcal{I}$ *be the set defined by* $1 \in \mathcal{I}$ *and* $\ell \in \mathcal{I}$ *for* $2 \leq \ell \leq L$ *if either the radius of convergence* R_{P_ℓ} *of* $P_\ell(z)$ *is less or equal to* $\tau^{[\ell-1]} > 1$ *or the corresponding* $\alpha_\ell =$

$\tau^{[\ell-1]}/P^{[\ell]}(\tau^{[\ell-1]}) < 1$. *For $T_L \to \infty$ so that $T_\ell = O(T)$ for $T \to \infty$, we have the asymptotic expansions*

$$B_{T_L}^{[L]}(1) = 1 - \frac{1}{1-P^{[L]'}(1)} \sum_{\ell \in \mathcal{I}} d_\ell P_\ell(\kappa_\ell)^{-(T_\ell-1)} - \frac{1}{1-P^{[L]'}(1)} \sum_{\ell \notin \mathcal{I}} O_1\Big(M_\ell T_\ell^{-3/2} r_\ell^{-T_\ell}\Big) + \sum_{\ell \in \mathcal{I}} O\Big(r_\ell^{-(T_\ell-1)}\Big) \tag{5.46}$$

$$B_{T_L}^{[L]'}(1) = \frac{1}{1-P^{[L]'}(1)} + \sum_{\ell \in \mathcal{I}} O\Big(T_\ell P_\ell(\kappa_\ell)^{-T_\ell}\Big) + \sum_{\ell \notin \mathcal{I}} O\Big(T_\ell^{-1/2} r_\ell^{-T_\ell}\Big)$$

with

$$d_\ell = \frac{\big(1-P^{[\ell]'}(1)\big)^2}{1-P^{[\ell-1]'}(1)} \cdot \frac{1-P_\ell(\kappa_\ell)P^{[\ell-1]'}(\kappa_\ell)}{P^{[\ell]'}(\kappa_\ell)-1} \cdot \frac{\kappa_\ell - 1}{\kappa_\ell} > 0 \tag{5.47}$$

$$r_\ell = P_\ell(\kappa_\ell) + \epsilon \qquad \text{for some } \epsilon > 0, \text{ if } \ell \in \mathcal{I} \text{ (with } r_\ell < \rho^{[\ell-1]} \text{ for } \ell \ge 2\text{)}, \tag{5.48}$$

where $P^{[0]}(z) \equiv 1$, $\kappa_\ell > 1$ denote the minimal solution of $x = P^{[\ell]}(x)$, $x > 1$, and

$$M_\ell = \frac{\big(1-P^{[\ell]'}(1)\big)^2}{1-P^{[\ell-1]'}(1)} \cdot \gcd(P^{[\ell-1]}) \cdot \sqrt{\frac{1}{2\pi P^{[\ell-1]}(\tau^{[\ell-1]})P^{[\ell-1]''}(\tau^{[\ell-1]})}} \cdot \left(\frac{1}{\tau^{[\ell-1]}\big(\rho^{[\ell-1]} - P_\ell(\tau^{[\ell-1]})\big)} - \frac{1}{\tau^{[\ell-1]}\big(\rho^{[\ell-1]} - p_{0,\ell}\big)} + \frac{P_\ell'(\tau^{[\ell-1]})\big(\tau^{[\ell-1]}-1\big)}{\big(\rho^{[\ell-1]} - P_\ell(\tau^{[\ell-1]})\big)^2}\right).$$

Proof. The recursive formulas provided by Lemma 5.6 and Lemma 5.7 express $V_{T_L}^{[L]}(.)$ in terms of $V_{T_{L-1}}^{[L-1]}(.)$ for $z \in \mathcal{D}(\alpha, \varepsilon)$. Since

$$V_{T_L}^{[L]}(z) = V_{T_L}^{[L]}\left(\frac{B^{[L]}(z)}{P^{[L]}\big(B^{[L]}(z)\big)}\right)$$

by (5.43), it turns out that we have to deal with a simple linear recurrence only: Abbreviating

$$V_\ell(z) = V_{T_\ell}^{[\ell]}\left(\frac{B^{[L]}(z)}{P^{[\ell]}\big(B^{[L]}(z)\big)}\right) + O\Big({V_{\zeta_\ell,T_\ell}^{[\ell]}}^2\Big) \tag{5.49}$$

it is not difficult to show that for $1 \le \ell \le L$

$$V_\ell(z) = C_\ell(z) V_{\ell-1}(z) + D_\ell(z), \tag{5.50}$$

with

$$C_\ell(z) = \frac{1 - \frac{B^{[L]}(z)}{P^{[\ell-1]}(B^{[L]}(z))} P^{[\ell-1]\prime}\big(B^{[L]}(z)\big)}{1 - \frac{B^{[L]}(z)}{P^{[\ell]}(B^{[L]}(z))} P^{[\ell]\prime}\big(B^{[L]}(z)\big)}, \quad \text{and}$$

$$\begin{aligned} D_\ell(z) = {} & -\delta_{\ell\in\mathcal{I}(\alpha)} \cdot \frac{1 - \frac{B^{[L]}(z)}{P^{[\ell]}(B^{[L]}(z))} P^{[\ell]\prime}\big(B^{[L]}(z)\big)}{1 - \frac{B^{[L]}(z)}{P^{[\ell-1]}(B^{[L]}(z))} P^{[\ell-1]\prime}\big(B^{[L]}(z)\big)} \\ & \cdot \frac{1 - \frac{B^{[L]}(z)}{P^{[\ell]}(B^{[L]}(z))} P_\ell\Big(K_\ell\big(B^{[L]}(z)\big)\Big) P^{[\ell-1]\prime}\Big(K_\ell\big(B^{[L]}(z)\big)\Big)}{1 - \frac{B^{[L]}(z)}{P^{[\ell]}(B^{[L]}(z))} P^{[\ell]\prime}\Big(K_\ell\big(B^{[L]}(z)\big)\Big)} \\ & \cdot \left(B^{[L]}(z) - \frac{B^{[L]}(z)^2}{K_\ell\big(B^{[L]}(z)\big)}\right)\left(\frac{P_\ell\Big(K_\ell\big(B^{[L]}(z)\big)\Big)}{P_\ell\big(B^{[L]}(z)\big)}\right)^{-(T_\ell-1)} \\ & + \delta_{\ell\in\mathcal{I}(\alpha)} O\left(\left(\frac{r_{\alpha',\ell}}{zP_\ell\big(B^{[L]}(z)\big)}\right)^{-(T_\ell-1)}\right) \\ & + \delta_{\ell\notin\mathcal{I}(\alpha)} O_1\left(M_\ell\left(\frac{B^{[L]}(z)}{P^{[\ell]}(B^{[L]}(z))}\right) T_\ell^{-3/2} \left(\frac{\rho^{[\ell-1]}}{zP_\ell\big(B^{[L]}(z)\big)}\right)^{-(T_\ell-1)}\right), \end{aligned} \tag{5.51}$$

where $\delta_{\ell\in\mathcal{I}(\alpha)} = 1$ if ℓ is such that $\alpha > \alpha_\ell$ and zero otherwise. For, considering (5.50) for $\ell = L$ yields the expansions provided by Lemmas 4.15 and 4.16, respectively. Note that, in the expression of Lemma 5.6,

$$O\left(\left(\frac{P_L\Big(K_L\big(B^{[L]}(z)\big)\Big)}{P_L\big(B^{[L]}(z)\big)}\right)^{-(T_L-1)} T_L V^{[L-1]}_{\kappa,T_{L-1}}\right) = O\left(\left(\frac{r_{\alpha,L}}{zP_L\big(B^{[L]}(z)\big)}\right)^{-T_L}\right) \tag{5.52}$$

since it can be shown by induction that $V^{[L-1]}_{\kappa,T_{L-1}}$ is exponentially small; recall that all $T_\ell = O(T)$ for $T \to \infty$ according to our assumptions on $\mathcal{T}_L$. Moreover, the remainder in (5.51) also hides the "artificial" $O\big({V^{[\ell]}_{\zeta_\ell,T_\ell}}^2\big)$ introduced in (5.49). Now, considering the expansions provided by Lemma 5.6 and Lemma 5.7 for $L-1$ and replacing z by $\zeta^{[L]}_\infty(z) = B^{[L]}(z)/P^{[L-1]}\big(B^{[L]}(z)\big)$, it is easy to see that this corresponds to a simple shift $L \to L-1$ of all instances of L except in $B^{[L]}(z)$, since $B^{[L-1]}\big(\zeta^{[L]}_\infty(z)\big) = B^{[L]}(z)$ and z appears in conjunction with $B^{[L]}(z)$ only. Note that a simple induction on (5.9) reveals

$$\zeta^{[L-k]}_\infty\Big(\cdots\zeta^{[L-1]}_\infty\big(\zeta^{[L]}_\infty(z)\big)\cdots\Big) = \frac{B^{[L]}(z)}{P^{[L-k-1]}\big(B^{[L]}(z)\big)}.$$

Hence, we may repeat this process of shifting and substitution for $\ell < L-1$, establishing the validity of the expressions $C_\ell(z)$ and $D_\ell(z)$.

Since $V_0 \equiv 0$ according to Lemma 5.6, the solution of the recurrence (5.50) is of course

$$V_L(z) = \sum_{\ell=1}^{L} \prod_{j=\ell+1}^{L} C_j(z) D_\ell(z); \tag{5.53}$$

note that

$$\prod_{j=\ell+1}^{L} C_j(z) = \frac{1 - \frac{B^{[L]}(z)}{P^{[\ell]}(B^{[L]}(z))} P^{[\ell]'}\Big(B^{[L]}(z)\Big)}{1 - \frac{B^{[L]}(z)}{P^{[L]}(B^{[L]}(z))} P^{[L]'}\Big(B^{[L]}(z)\Big)}.$$

Remembering the values (5.2) given in Lemma 5.1, substituting $z = 1$ in the expressions for $\prod_j C_j(z)$ and $D_\ell(z)$ above easily provides the values

$$\prod_{j=\ell+1}^{L} C_j(1) = \frac{1 - P^{[\ell]'}(1)}{1 - P^{[L]'}(1)}$$

$$D_\ell(1) = \delta_{\ell\in\mathcal{I}} \frac{1 - P^{[\ell]'}(1)}{1 - P^{[\ell-1]'}(1)} \cdot \frac{1 - P_\ell(\kappa_\ell) P^{[\ell-1]'}(\kappa_\ell)}{P^{[\ell]'}(\kappa_\ell) - 1} \left(\frac{\kappa_\ell - 1}{\kappa_\ell}\right) P_\ell(\kappa_\ell)^{-(T_\ell-1)}$$
$$+ \delta_{\ell\in\mathcal{I}} O\Big(r_\ell^{-(T_\ell-1)}\Big) + \delta_{\ell\notin\mathcal{I}} O_1\Big(M_\ell T_\ell^{-3/2} (\rho^{[\ell-1]})^{-(T_\ell-1)}\Big)$$

where $\mathcal{I} = \mathcal{I}(1)$ and $M_\ell = -\big(1 - P^{[\ell]'}(1)\big) M_\ell(1)$, cf. (5.45). Plugging this into (5.53) while recalling $B_{T_L}^{[L]}(z) = B^{[L]}(z) - V_{T_L}^{[L]}(z)$, (5.46) and (5.47) follow. To confirm that $d_\ell > 0$, we note that $P^{[\ell-1]'}(1) < P^{[\ell]'}(1) < 1$ by (2.4) and $\kappa_\ell > 1$ by (5.2), so that the first and third factor of (5.47) are positive. To show that this is also true for the second one, we exploit

$$\frac{1 - P_\ell(\kappa_\ell) P^{[\ell-1]'}(\kappa_\ell)}{P^{[\ell]'}(\kappa_\ell) - 1} = -\frac{1}{1 - U^{[\ell]'}\Big(\kappa_\infty^{[\ell]}(1)\Big)}$$

by virtue of (5.42). Since $\kappa_\infty^{[\ell]}(z)$ is the 2nd solution of (5.6), which exists since $\ell \in \mathcal{I}$, expression (4.7) for $\alpha = 1$ reveals that the right hand side of the equation above is indeed > 0.

Finally, since the remainder terms in our expansion represent analytic functions and are uniformly valid in a complex neighborhood of $\alpha = 1$, differentiation of (5.53) is permitted and the value for $B_{T_L}^{[L]'}(1) = B^{[L]'}(1) - V_{T_L}^{[L]'}(1)$ follows easily by using the values given in (5.2). Note that we employed coarse bounds on the terms involved only. This eventually completes the proof of Lemma 5.8. □

Now we are ready for our final theorem:

Theorem 5.9. *With the conditions from §2, there is some $\epsilon > 0$ such that the successful run duration S_{T_L} for static priority scheduling with $L \geq 1$ priority levels is approximately exponentially distributed with parameter $1/\mu_{T_L}^{[L]}$ satisfying*

$$\mu_{T_L}^{[L]} = \frac{1}{\sum_{\ell\in\mathcal{I}} d_\ell P_\ell(\kappa_\ell)^{-(T_\ell-1)} + \sum_{\ell\notin\mathcal{I}} O_1\Big(M_\ell T_\ell^{-3/2} (\rho^{[\ell-1]})^{-(T_\ell-1)}\Big)} \cdot \Big(1 + O\big((1+\epsilon)^{-T}\big)\Big) \tag{5.54}$$

for $T_L \to \infty$ in a way that ensures $T_\ell = O(T)$ for $T \to \infty$.

Herein, $\mathcal{I}$ is the set defined by $1 \in \mathcal{I}$ and $\ell \in \mathcal{I}$ for $2 \leq \ell \leq L$ if either the radius of convergence R_{P_ℓ} of $P_\ell(z)$ is less or equal to $\tau^{[\ell-1]} > 1$ or $\tau^{[\ell-1]}/P^{[\ell]}(\tau^{[\ell-1]}) < 1$, where $\tau^{[\ell]}$

denotes the minimal positive solution of $xP^{[\ell]\prime}(x) - P^{[\ell]}(x) = 0$ *for* $P^{[\ell]}(z) = \prod_{j=1}^{\ell} P_j(z)$. *In addition,*

$$P_\ell(\kappa_\ell) > 1 \qquad \text{with } \kappa_\ell > 1 \text{ being the minimal solution of } x = P^{[\ell]}(x) \text{ for } x > 1,$$

$$d_\ell = \frac{\big(1 - P^{[\ell]\prime}(1)\big)^2}{1 - P^{[\ell-1]\prime}(1)} \cdot \frac{1 - P_\ell(\kappa_\ell)P^{[\ell-1]\prime}(\kappa_\ell)}{P^{[\ell]\prime}(\kappa_\ell) - 1} \cdot \frac{\kappa_\ell - 1}{\kappa_\ell} > 0$$

with $P^{[0]}(z) \equiv 1$. *Finally,* O_1 *denotes a O-term with implied constant* $M \le 1 + \epsilon_O$, *for some small* $\epsilon_O \ge 0$,

$$M_\ell = \frac{\big(1 - P^{[\ell]\prime}(1)\big)^2}{1 - P^{[\ell-1]\prime}(1)} \cdot \gcd(P^{[\ell-1]}) \cdot \sqrt{\frac{1}{2\pi P^{[\ell-1]}(\tau^{[\ell-1]})P^{[\ell-1]\prime\prime}(\tau^{[\ell-1]})}} \cdot \left(\frac{1}{\tau^{[\ell-1]}\big(\rho^{[\ell-1]} - P_\ell(\tau^{[\ell-1]})\big)} - \frac{1}{\tau^{[\ell-1]}\big(\rho^{[\ell-1]} - p_{0,\ell}\big)} + \frac{P_\ell'(\tau^{[\ell-1]})\big(\tau^{[\ell-1]} - 1\big)}{\big(\rho^{[\ell-1]} - P_\ell(\tau^{[\ell-1]})\big)^2}\right) > 0,$$

and $\rho^{[\ell]} = \tau^{[\ell]}/P^{[\ell]}(\tau^{[\ell]}) > 1$.

With the additional assumption that the greatest common divisor of the non-zero-coefficient indices ($n \ge 1$) *in the Taylor expansion of* $P^{[L]}(z)$ *satisfies* $\gcd(P^{[L]}) = 1$, *there is some* $\delta > 0$ *such that*

$$v_{n,\mathcal{T}_L} = \boldsymbol{P}\{\text{successful run duration } \mathcal{S}_{\mathcal{T}_L} \text{ has length} \le n\}$$

may be expressed as

$$v_{n,\mathcal{T}_L} = 1 - \big(1 + O(1/\mu_{\mathcal{T}_L}^{[L]})\big)e^{-\mu_{\mathcal{T}_L}^{[L]-1}(1+O(1/\mu_{\mathcal{T}_L}^{[L]}))n} + O\big(\mu_{\mathcal{T}_L}^{[L]-1}(1+\delta)^{-n}\big) \tag{5.55}$$

for $\mathcal{T}_L \to \infty$ *sufficiently large, uniformly valid for* arbitrary $n \ge 1$. *The* m*-th moment* $E[\mathcal{S}_{\mathcal{T}_L}^m]$ *of* $\mathcal{S}_{\mathcal{T}_L}$ *fulfills*

$$E[\mathcal{S}_{\mathcal{T}_L}^m] = m!\big[\mu_{\mathcal{T}_L}^{[L]}\big(1 + O(1/\mu_{\mathcal{T}_L}^{[L]})\big)\big]^m + O\left(\mu_{\mathcal{T}_L}^{[L]-1}\frac{m!\sqrt{m}}{(2\pi e\delta)^m}\right) \tag{5.56}$$

for $\mathcal{T}_L \to \infty$ *sufficiently large, which is again uniformly valid for* arbitrary $m \ge 1$. *Note that the sign of the first O-term is positive.*

Proof. We first have to evaluate (2.5). Abbreviating the major (first) factor in (5.54) by $\overline{\mu}_{\mathcal{T}_L}^{[L]}$, applying $1/\big(1 + O(x)\big) = 1 + O(x)$ for $x \to 0$ to the major expressions of Lemma 5.8 plugged in (2.5) provides

$$\mu_{\mathcal{T}_L}^{[L]} = \overline{\mu}_{\mathcal{T}_L}^{[L]}\left(1 + \frac{\sum_{\ell\in\mathcal{I}} O\big(r_\ell^{-(T_\ell-1)}\big)}{\sum_{\ell\in\mathcal{I}} d_\ell P_\ell(\kappa_\ell)^{-(T_\ell-1)} + \sum_{\ell\notin\mathcal{I}} O_1\big(M_\ell T_\ell^{-3/2} r_\ell^{-(T_\ell-1)}\big)}\right).$$

Recalling (5.48), it is immediately apparent that the fraction above is exponentially small in T; note that at least $1 \in \mathcal{I}$.

All preconditions required for applying the results of [DS93] follow immediately from our conditions in §2, except that we have to add the additional restriction $\gcd(P^{[L]}) = 1$ here. The limiting distribution is thus approximately exponential with rate $1/\mu_{\mathcal{T}_L}^{[L]}$, and Theorems 3.5 and 3.7 of [DS93] immediately provide the uniform asymptotic expansions for probabilities and moments. The positive sign of the first O-term in the expansion of the moments follows from the remark on p. 16 of [DS93]. □

Note that it is possible to remove some of the multiple O-terms in the expansions for probabilities and moments in Theorem 5.9. For example, if m is fixed, we clearly have $E[\mathcal{S}_{\mathcal{T}_L}^m] = m!(\mu_{\mathcal{T}_L}^{[L]})^m\big(1 + O(1/\mu_{\mathcal{T}_L}^{[L]})\big)$. However, this would necessarily impair the general uniformity of our formulas w.r.t. n resp. m. Therefore, we prefered to provide all information required for evaluating the expansions for a particular ratio of n resp. m versus $\mathcal{T}_L$.

We conclude this section with a few remarks on the asymptotic formulas in Theorem 5.9. First of all, it should be clear that $\mu_{\mathcal{T}_L}^{[L]}$ grows exponentially with the deadlines. To justify why we included the O_1-term in the major contributions of (5.54), we should mention that dealing with the contributions of a priority level $\ell \notin \mathcal{I}$ was not considered very important initially. However, when conducting our numerical examples, we recognized that the fraction of conceivable operating conditions that were covered by our original formulas was less than expected (see our remarks in §6). Looking at a way to extend the range of applicability, i.e., improving the remainder terms, we realized that a full treatment would require additional tools (dealing with the solutions of a bivariate functional equation at singular points), thus making a very long paper even much longer. Hence we decided to get additional information out of our analysis without too much effort. The O_1-term was found to be a convenient way of doing this: It disappears in our original setting and provides a (usually tight) upper bound for other operating conditions.

6. Conclusions

In this paper, we quantified deadline meeting properties of the widespreadly used static priority scheduling algorithm (SPS) employed for scheduling probabilistically arriving tasks in real-time systems. More specifically, we determined the distribution of the duration $\mathcal{S}_{\mathcal{T}_L}$ a discrete-time single server system using SPS operates without violating any tasks deadline. This distribution is known to be asymptotically exponential with some parameter $\lambda_{\mathcal{T}_L} = 1/\mu_{\mathcal{T}_L}^{[L]}$ for any scheduling algorithm. Hence, we had to provide an asymptotic formula for $\mu_{\mathcal{T}_L}^{[L]} = \mathrm{E}[\mathcal{S}_{\mathcal{T}_L}]$ to arrive at the probabilities and moments given in Theorem 5.9.

Apart from the fact that deadline meeting capabilities of different scheduling algorithms may be compared via the distribution of $\mathcal{S}_{\mathcal{T}_L}$, i.e., via the value of $\mu_{\mathcal{T}_L}^{[L]}$, our results provide an answer to the following practical question: Given the input probability distributions for a certain (high-)load situation, and a (tolerable) probability p for deadline missing (say, $p = 10^{-6}$), what is the maximum duration such a situation may last in order to guarantee p? It is immediately apparent from plugging in $n = p\mu_{\mathcal{T}_L}^{[L]}$ into the expression for $v_{n,\mathcal{T}_L}$ in Theorem 5.9 that the probability of no deadline violation during the next $p\mu_{\mathcal{T}_L}^{[L]}$ cycles is $1 - v_{n,\mathcal{T}_L} = p + O(p^2)$. Since $\mu_{\mathcal{T}_L}^{[L]}$ increases exponentially with the deadline(s), such systems could be expected to operate properly a long time.

In order to answer the question whether and when this expectation is justified (and to get a feeling what can be done with our results), we computed several numerical examples. More specifically, we considered tasks at $L = 3$ priority levels, described by

- Poisson arrival PGF's $A_\ell(z) = e^{\lambda_\ell(z-1)}$ with average arrival rate $A'_\ell(1) = \lambda_\ell$, $1 \le \ell \le 3$,

- task execution time PGF's $L_\ell(z) = z^{l_\ell}e^{l_\ell(z-1)/R}$ with average task execution time $L'_\ell(1) = l_\ell(1+R)/R$, for some (fixed) "randomness parameter" $R \leq \infty$,
- deadlines $T_\ell = K_\ell \cdot l_\ell$.

Several different scenarios were evaluated by a number of MAPLE-programs that compute the major factor $\log \overline{\mu}_\mathcal{T}$ of (5.54)—with $\mathcal{T} = \{T_1, T_2, T_3\}$ denoting the set of *relative deadlines* here— as a function of the parameters involved. More specifically, we considered

(1) $l_\ell = E_\ell \cdot 1000$, $R = 10$,
(2) $l_\ell = E_\ell \cdot 10$, $R = 10$,
(3) $l_\ell = E_\ell \cdot 1100$, $R = \infty$

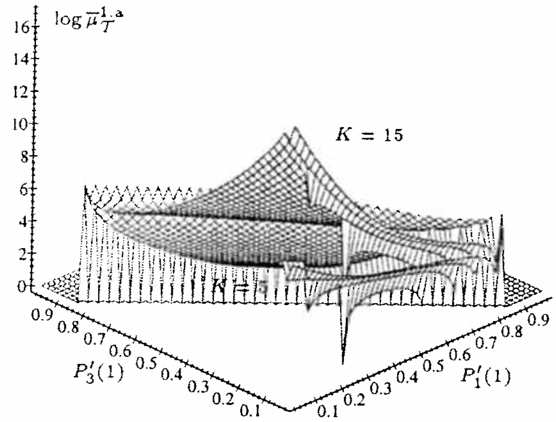
with $\mathcal{E} = \{E_1, E_2, E_3\}$ denoting the set of *relative task execution times* for a number of different settings of the other parameters. Note, however, that all scenarios above are "compatible": scenario 2 can be considered as a less preemptible version of 1 with cycles being 100 times larger, and scenario 3 is a less random version of 1 w.r.t. task execution times, providing $\gcd(P_\ell) = 100$ as well. All scenarios were made further comparable by varying the average overall execution time $P'_\ell(1) = A'_\ell(1)L'_\ell(1)$ of level-ℓ tasks, which is always in the interval $0, \ldots, 1$, instead of the arrival rate.

Fixing $P'_2(1) = 0.15\left[\frac{\text{actions}}{\text{cycle}}\right]$ and $E_2 = 2$ for all settings, we computed 3D-plots of $\log \overline{\mu}_\mathcal{T} = \log \overline{\mu}_\mathcal{T}(x, y)$ as a function of $x = P'_1(1)$, $y = P'_3(1)$ for

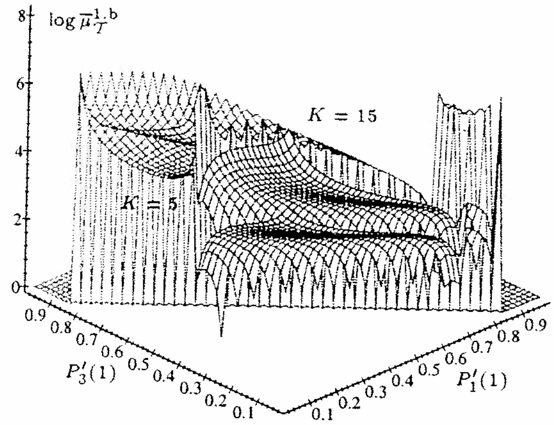
(a) $E_1 = 1$, $E_3 = 4$,
(b) $E_1 = 4$, $E_3 = 1$,
(c) $E_1 = 2$, $E_3 = 2$

with $K_1 = K_2 = K_3 = K \in \{2, 5, 10, 20\}$. We provided the plots for all scenarios in the appendix and include the ones for scenario 1.a below for the ease of reference. Moreover, comparative plots providing $\overline{\mu}_\mathcal{T}^{1.y}/(100\overline{\mu}_\mathcal{T}^{2.y})$ and $\overline{\mu}_\mathcal{T}^{1.y}/\overline{\mu}_\mathcal{T}^{3.y}$ for relative deadlines $\mathcal{T}_3 = \{K1, K2, K3\} = \{5, 5, 5\}$ are also appended. They show that changing randomness of task execution times (R) and preemption granularity both have minor effect upon our results, at least in all cases where our results fully apply ($\mathcal{I} = \{1, \ldots, L\}$).

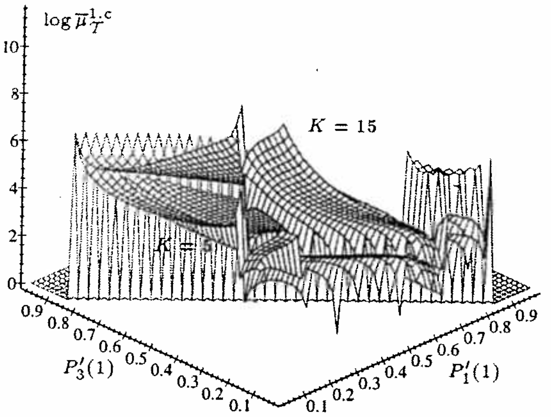
From plots 1.a–1.c provided below it is apparent that the most regular behavior is obtained in 1.a. In this case, the solution κ_3 exists such that $3 \in \mathcal{I}$ ($\alpha_3 < 1$). The more irregular behavior in 1.b and 1.c arises from the fact that a (very) low-load task does (almost) not contribute to $\overline{\mu}_{\mathcal{T}_3}$ if put on level 3, irrespectively of its deadline. This results in $3 \notin \mathcal{I}$ ($\alpha_3 > 1$) for small $P'_3(1)$, i.e., the case where the O_1-term in (5.54) comes into play. The ditch clearly visible in our plots marks the border $\alpha_3 = 1$, where our analysis is not applicable. Note that it appears at a higher load in plot 1.c and 1.b due to the longer task execution times of the level-3 task. However, it is clear that the ditch is an artefact of our analysis that fails to approximate the real (smooth) contour in this case. The O_1-term responsible for handling $\alpha_3 > 1$ does a very good job in approximating the low-load behavior for scenario 1 and 2, where $\gcd(P^{[2]}) = 1$. However, as can be seen from the plots 3.a–3.c (noD) in the appendix, it is about 100 times too large for scenario 3 where $\gcd(P^{[2]}) = 100$ (which is not at all surprising).



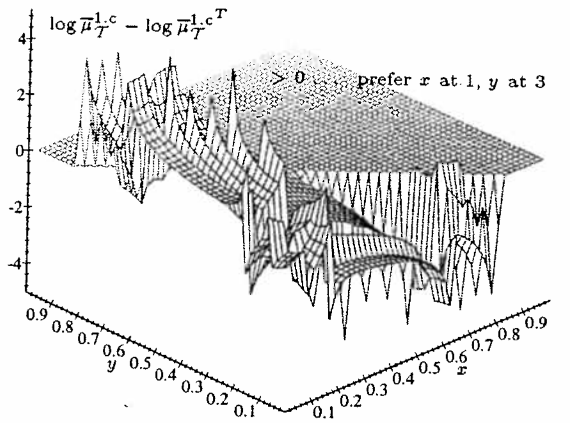
Plot scenario 1.a
rel. task exec times $\mathcal{E} = \{1, 2, 4\}$
rel. deadlines $\mathcal{T} = \{K, K, K\}$



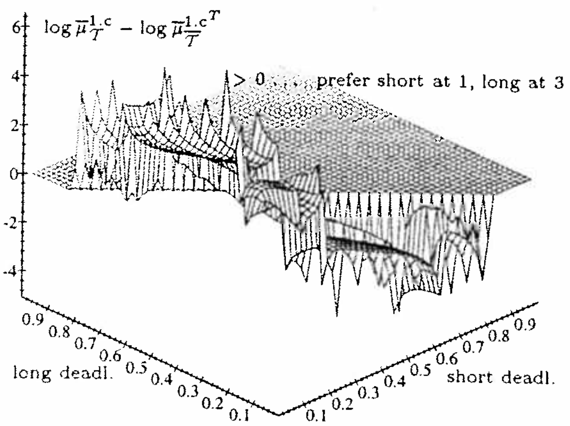
Plot scenario 1.b
rel. task exec times $\mathcal{E} = \{4, 2, 1\}$
rel. deadlines $\mathcal{T} = \{K, K, K\}$



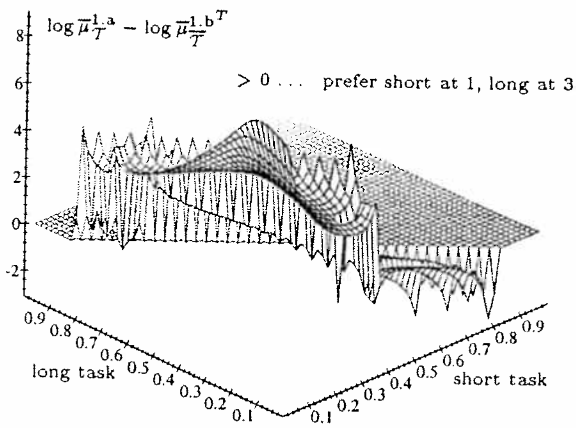
Plot scenario 1.c
rel. task exec times $\mathcal{E} = \{2, 2, 2\}$
rel. deadlines $\mathcal{T} = \{K, K, K\}$



Plot high load vs. low load (1.c.i)
rel. task exec times $\mathcal{E} = \{2, 2, 2\}$
rel. deadlines $\mathcal{T} = \{5, 5, 5\}$



Plot short vs. long deadline (1.c.ii)
rel. task exec times $\mathcal{E} = \{2, 2, 2\}$
rel. deadlines $\mathcal{T} = \{5, 10, 15\}$, $\overline{\mathcal{T}} = \{15, 10, 5\}$



Plot short vs. long task exec time (1.a/b.iii)
rel. task exec times $\mathcal{E} = \{1, 2, 4\}$, $\overline{\mathcal{E}} = \{4, 2, 1\}$
rel. deadlines $\mathcal{T} = \{5, 10, 15\}$, $\overline{\mathcal{T}} = \{15, 10, 5\}$

Finally, a number of plots of differences of $\log \overline{\mu}_T^{1,y}$ for certain parameter settings were computed. They answer the question whether a task should be put at level 1 or level 3 w.r.t. better deadline meeting properties. We considered 3 different questions here:

(i) High load vs. low load for tasks with same task execution time ($\mathcal{E} = \{2,2,2\}$) and same deadline ($\mathcal{T}_3 = \{5,5,5\}$): Plot $\Delta(x,y) = \log \overline{\mu}_T^{1,c}(x,y) - \log \overline{\mu}_T^{1,c}(y,x)$, which of course satisfies $\Delta(x,y) = -\Delta(y,x)$. If $\Delta(x,y) > 0$, then the task with load x should be put at level 1 and the one with load y at level 3, and vice versa otherwise.

(ii) Short deadline vs. long deadline for tasks with same task execution time ($\mathcal{E} = \{2,2,2\}$): Plot $\Delta(x,y) = \log \overline{\mu}_T^{1,c}(x,y) - \log \overline{\mu}_{\overline{T}}^{1,c}(y,x)$, where $\mathcal{T} = \{5,10,15\}$ resp. $\overline{\mathcal{T}} = \{15,10,5\}$. Hence, x resp. y is the arrival load for the task with the short resp. long deadline. If $\Delta(x,y) > 0$, then the task with the short deadline should be put at level 1.

(iii) Short task ($\mathcal{E} = \{1,2,4\}$) vs. long task ($\overline{\mathcal{E}} = \{4,2,1\}$) with short task has narrowest relative deadline ($\mathcal{T} = \{2,5,10\}$: Plot $\Delta(x,y) = \log \overline{\mu}_T^{1,a}(x,y) - \log \overline{\mu}_{\overline{T}}^{1,b}(y,x)$, so that x resp. y is the arrival load for the short resp. the long task. If $\Delta(x,y) > 0$, then the short task should be put at level 1.
We provide three different settings of deadlines here: shortest task has narrowest relative deadline ($\mathcal{T} = \{5,10,15\}$ and $\overline{\mathcal{T}} = \{15,10,5\}$), all tasks have same relative deadline ($\mathcal{T} = \overline{\mathcal{T}} = \{5,5,5\}$), and shortest task has widest deadline ($\mathcal{T} = \{15,10,5\}$ and $\overline{\mathcal{T}} = \{5,10,15\}$).

In summary, our examples show that large $\mu_{T_L}^{[L]}$ —and hence reasonable deadline meeting properties— can be expected for low load and, in particular, relatively large deadlines only. Most real-time system designers are used to be quite generous when dimensioning computing resources, which may be the reason that most contemporary systems work reasonably well in practice. However, our results show that one has to be very careful when relaxing apparently overly conservative assumptions.

There are several directions of further research linked up with this paper. Most importantly, we are considering the question how to generalize our input model to allow for time-varying and not independent input distributions, a situation quite common in real-time systems. Moreover, there are of course other scheduling algorithms for real-time systems that should analyzed by means of our approach. In particular, there are the earliest deadline first algorithm —a really tough problem— and a whole family of scheduling algorithms designed for working in conjunction with deterministic (hard real-time) schedulers. A minor —but relevant and not at all trivial— extension of SPS would be to consider several tasks with different deadlines at the same priority level.

We think that our general approach has proven its appropriateness in providing results that have not been known before with manageable effort. Actually, we do not know of any other research that provides results that are compareable to ours. The conceptual issues underlying our derivations are reasonably simple and supported by powerful general results. The analysis in this paper, which is admittedly long, touches surprisingly mathematical problems that are interesting in their own right, so we thought it appropriate to provide all the technical details.

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Glossary

Name	Meaning	Def.	Pg.
$A_\ell(z) = \sum_{k\geq 0} a_{k,\ell} z^k$	PGF task arrivals level ℓ	(2.2)	2
α_l	critical value for 2nd solution	Lemma 5.3	29
$B^{[L]}(z) = \sum_{n\geq 0} b_n^{[L]} z^n$	PGF unrestricted busy periods	(3.3)	5
$B_{\mathcal{T}_L}^{[L]}(z) = \sum_{n\geq 0} b_{n,\mathcal{T}_L}^{[L]} z^n$	PGF $\mathcal{T}_L$-feasible busy periods	Theorem 3.1	11
$B_{\mathcal{T}_{L-1}}^{[L-1]}(z)$	PGF $\mathcal{T}_{L-1}$-feasible busy periods	Theorem 3.1	11
β	coefficient in solution's expansion	(4.4)	15
$\beta^{[L]}$	coefficient in expansion $B^{[L]}(z)$	Lemma 5.1	27
d	gcd indices non-zero coefficients in $U(s)$	Lemma 4.4	19
$d^{[L]}$	gcd indices in $P^{[L]}(z)$	Lemma 5.1	27
$\mathcal{D}(z_0, R)$	open disk with radius R around z_0	§4	13
$\overline{\mathcal{D}}(z_0, R)$	closed disk with radius R around z_0	§4	13

Name	Meaning	Def.	Pg.
$\Delta_\rho = \Delta_\rho(\eta, \varphi, d)$	*indented* disk sparing out $\rho e^{2\pi i l/d}$	§4	13
δ	coefficient in solution's expansion	(4.6)	15
$F(s,z) = zU(s) - s = 0$	general functional equation	(4.1)	14
γ	coefficient in solution's expansion	(4.5)	15
$\gcd(f)$	gcd indices non-zero coefficients in $f(z)$	Theorem 3.1	11
$K^{[L]}(z)$	2nd solution of $zP^{[L]}(s) - s = 0$	Lemma 5.1	27
$\kappa_L = K^{[L]}(1)$	value 2nd solution of $zP^{[L]}(s) - s = 0$	Lemma 5.1	27
$\kappa(z)$	general 2nd solution	Lemma 4.2	16
$\kappa_\infty^{[L]}$	$\kappa(z)$ for $zP_L\left(B^{[L-1]}(s)\right) - s = 0$	Lemma 5.3	29
$\kappa_{\mathcal{T}_{L-1}}^{[L]}$	$\kappa(z)$ for $zP_L\left(B_{\mathcal{T}_{L-1}}^{[L-1]}(s)\right) - s = 0$	Lemma 5.4	32
L	lowest priority (= no. of priority levels)	§2	2
$L_\ell(z) = \sum_{k\geq 1} l_{k,\ell} z^k$	PGF task execution time level ℓ	(2.1)	2
$\mu_{\mathcal{T}_L}$	expected length $\mathcal{T}_L$-feasible busy period	(2.5)	4
$M_{\alpha,L}(z)$	implied "constant" final O_1-term	Lemma 5.7	43
O_1	O-term with implied constant ≈ 1	Lemma 4.7	23
$P_\ell(z) = \sum_{k\geq 0} p_{k,\ell} z^k$	PGF overall execution time level ℓ	(2.3)	2
$p_{0,\ell} = P_\ell(0)$	**P** no arrival at level-ℓ	(2.3)	2
$P^{[L]}(z) = \prod_{\ell=1}^{L} P_\ell(z)$	cumulated overall exec. time levels $1,\ldots,L$	(3.2)	5
r, r_α	radius of solutions' closed s-domain	Lemma 4.2	16
$r_{\alpha,L}$	remainder term $B_{\mathcal{T}_L}^{[L]}(z)$	Lemma 5.5	36
R_U	radius of convergence $U(s)$	(4.1)	14
$\rho = \frac{\tau}{U(\tau)}$	general branch point of solutions	(4.3)	14
$\rho_l = \rho e^{2\pi i l/d}$	general complex branch points	Lemma 4.4	19
$\rho^{[L]}$	ρ for $zP^{[L]}(s) - s = 0$	Lemma 5.1	27
$\rho_\infty^{[L]}$	ρ for $zP_L\left(B^{[L-1]}(s)\right) - s = 0$	Lemma 5.3	29
$\rho_{\mathcal{T}_{L-1}}^{[L]}$	ρ for $zP_L\left(B_{\mathcal{T}_{L-1}}^{[L-1]}(s)\right) - s = 0$	Lemma 5.4	32
$\mathcal{T}_L = \{T_L, \ldots, T_1\}$	set of all deadlines	§2	3
$\mathcal{T}_{L-1} = \{T_{L-1}, \ldots, T_1\}$	set of all deadlines except priority L	§2	3
τ	general solution of $xU'(x) - U(x) = 0$	(4.2)	14
$\tau^{[L]}$	τ for $zP^{[L]}(s) - s = 0$	Lemma 5.1	27
$\tau_\infty^{[L]}$	τ for $zP_L\left(B^{[L-1]}(s)\right) - s = 0$	Lemma 5.3	29
$\tau_{\mathcal{T}_{L-1}}^{[L]}$	τ for $zP_L\left(B_{\mathcal{T}_{L-1}}^{[L-1]}(s)\right) - s = 0$	Lemma 5.4	32
$U(s)$	general generator function in $F(s,z) = 0$	(4.1)	14
$V_{\mathcal{T}_L}^{[L]}(z) = B^{[L]}(z) - B_{\mathcal{T}_L}^{[L]}(z)$	Diff. PGF feasible/arbitrary busy periods	Lemma 5.2	28
$\zeta(z)$	general 1st ("natural") solution	Lemma 4.2	16
$\zeta_\infty^{[L]}$	$\zeta(z)$ for $zP_L\left(B^{[L-1]}(s)\right) - s = 0$	Lemma 5.3	29
$\zeta_{\mathcal{T}_{L-1}}^{[L]}$	$\zeta(z)$ for $zP_L\left(B_{\mathcal{T}_{L-1}}^{[L-1]}(s)\right) - s = 0$	Lemma 5.4	32

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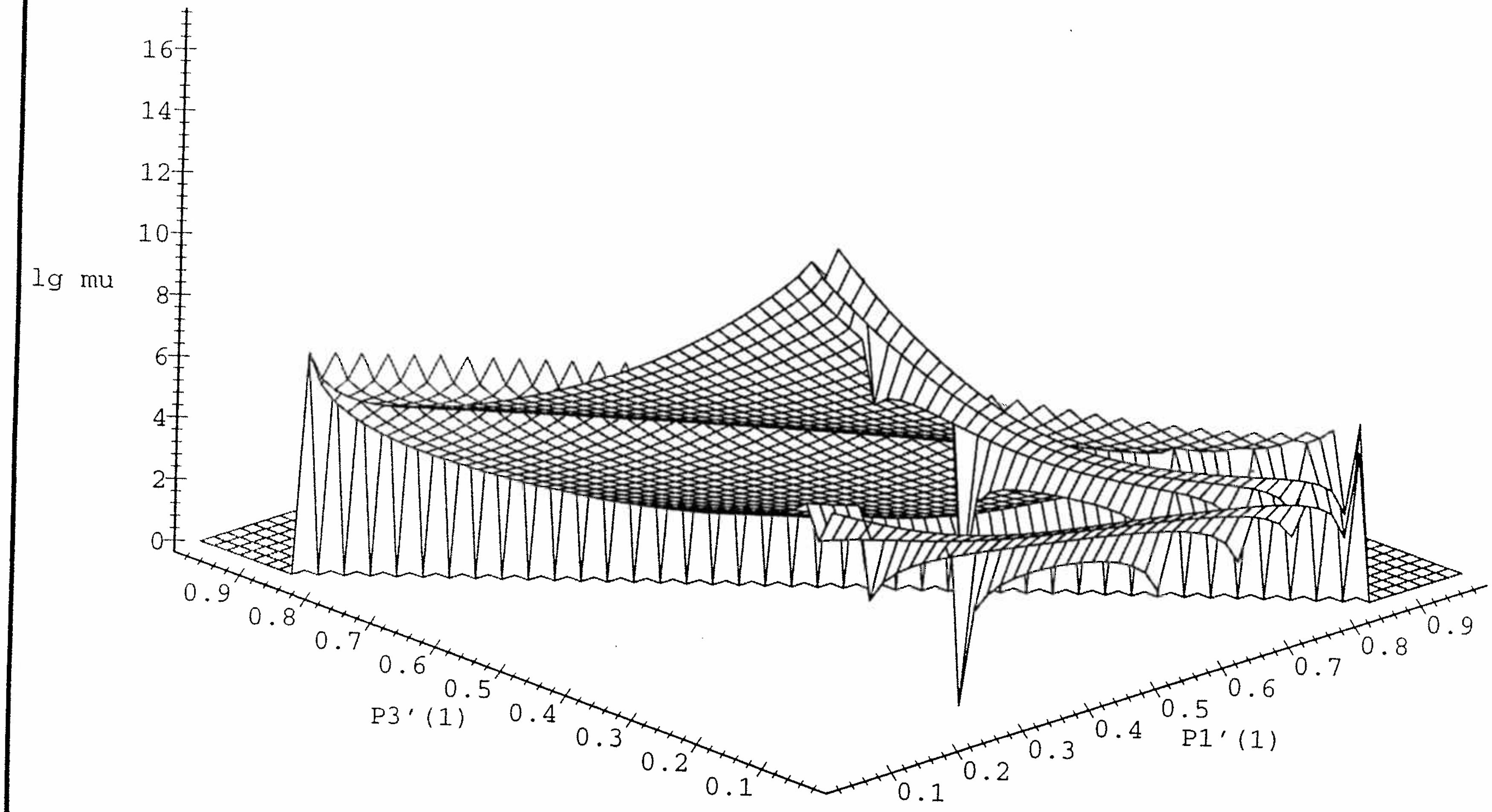
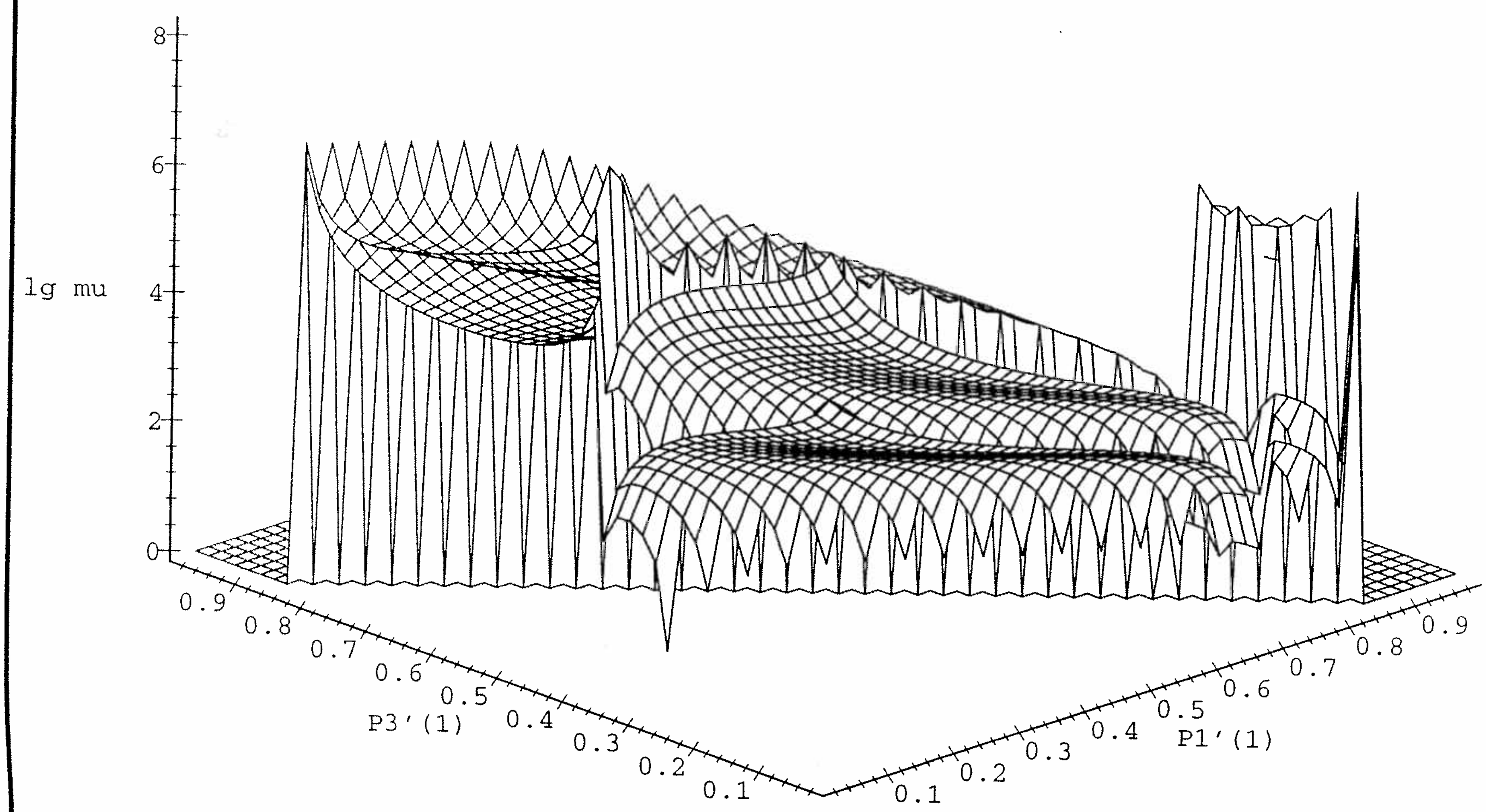
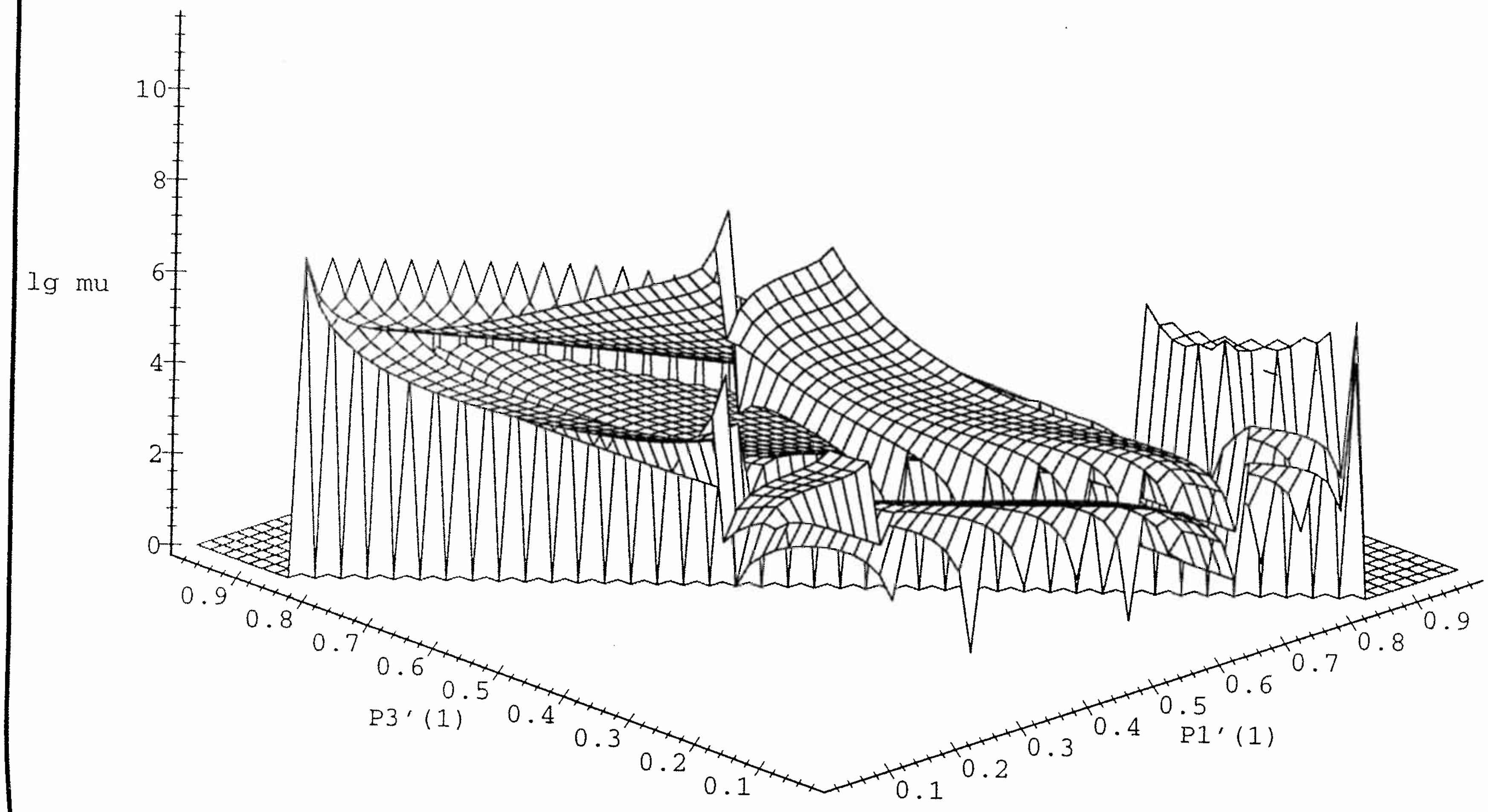
Plot 1.a: E = {1,2,4}, T = {5,5,5}



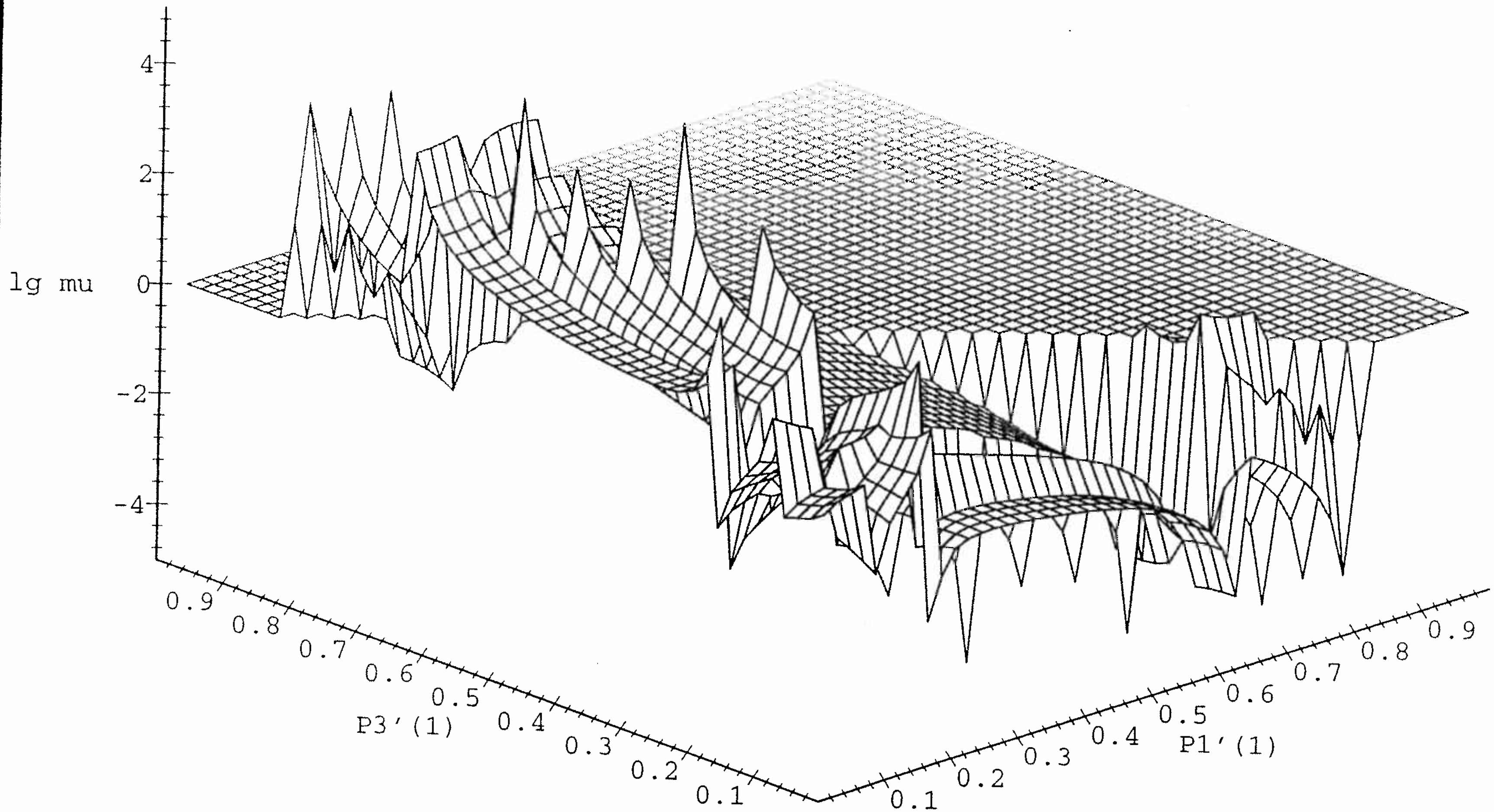
Plot 1.b: $E = \{4,2,1\}$, $T = \{5,5,5\}$



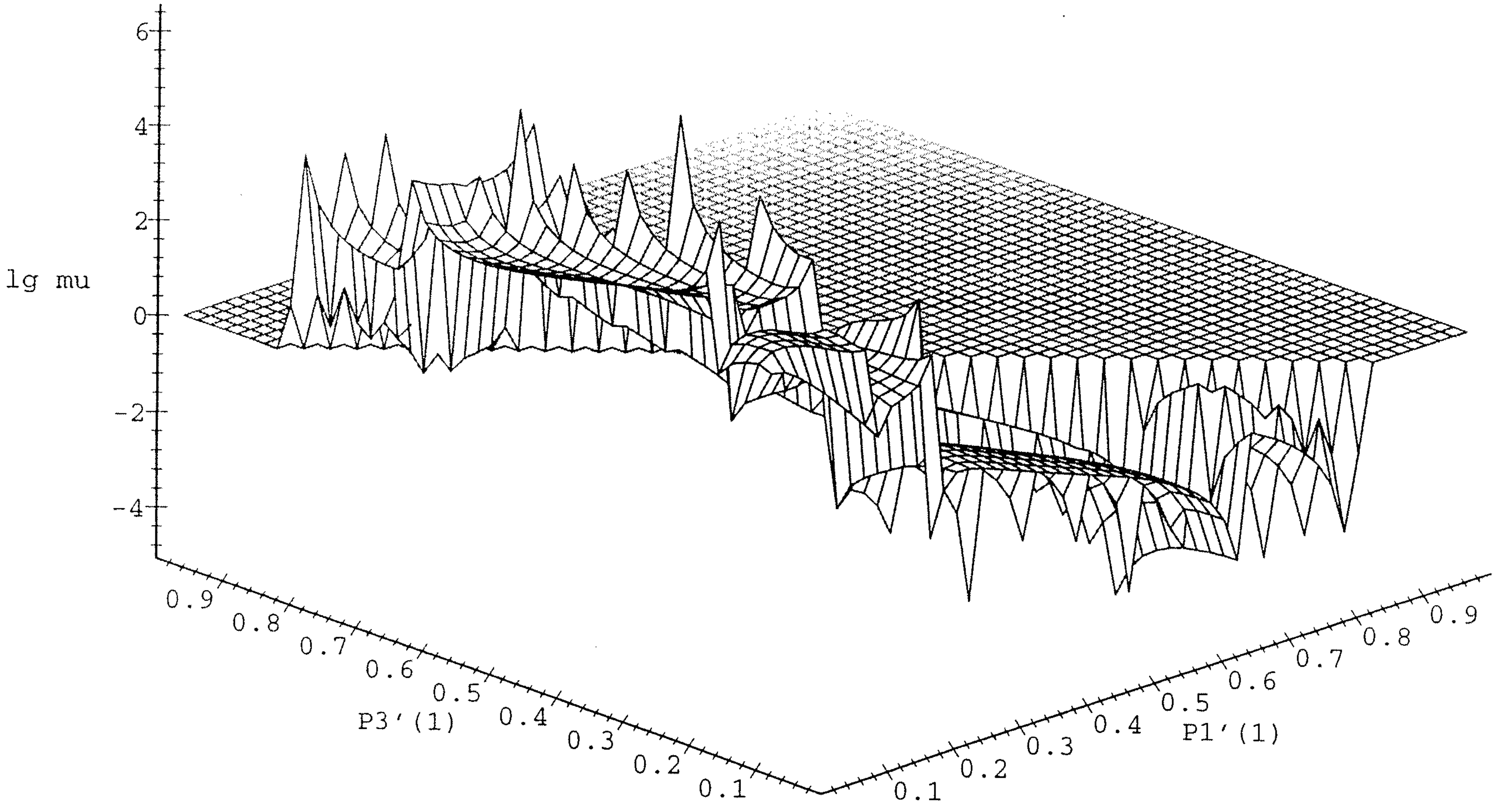
Plot 1.c: E = {2,2,2}, T = {5,5,5}



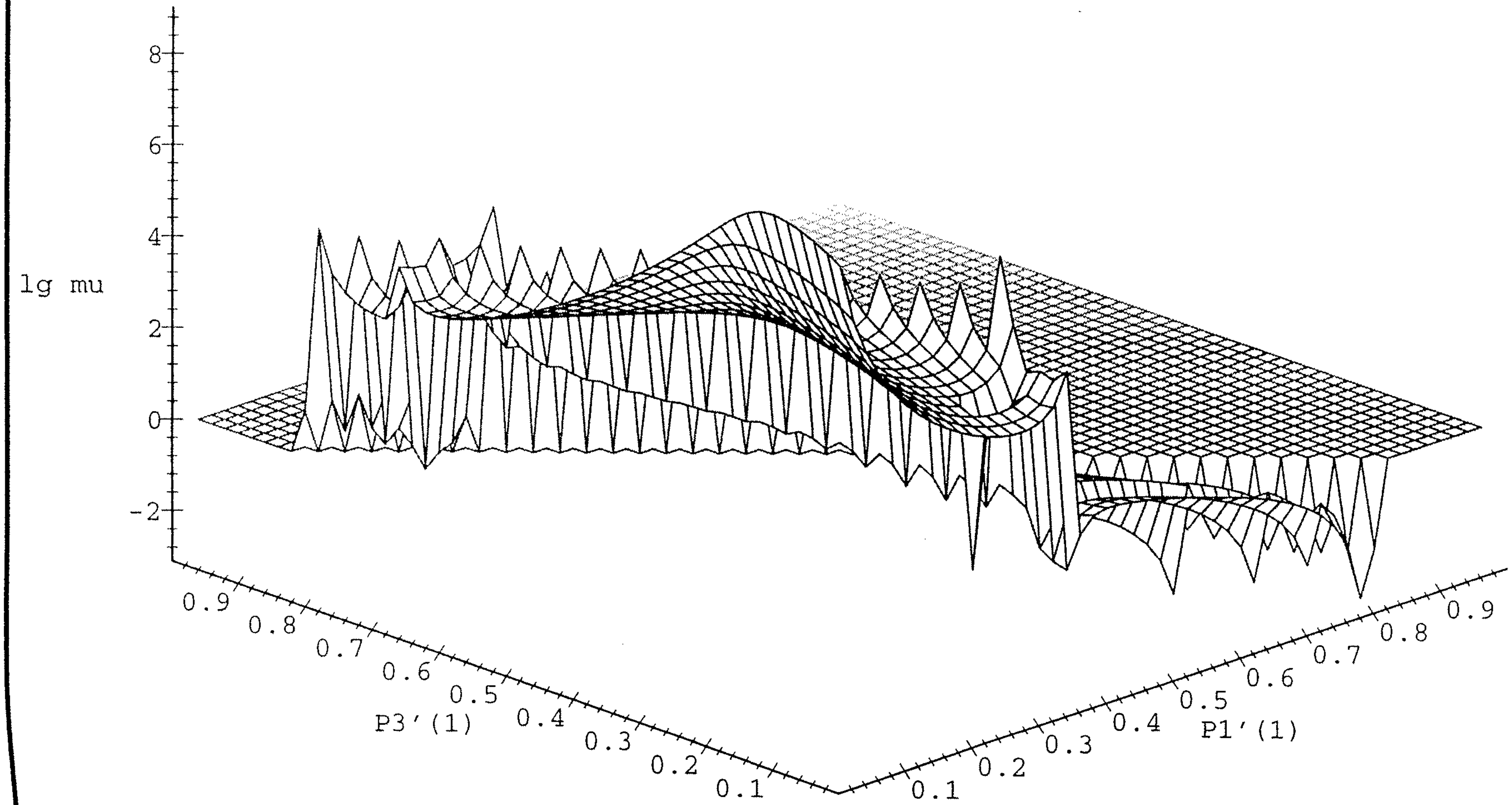
Plot 1.c-1.cT: E = {2,2,2}, T = {5,5,5}



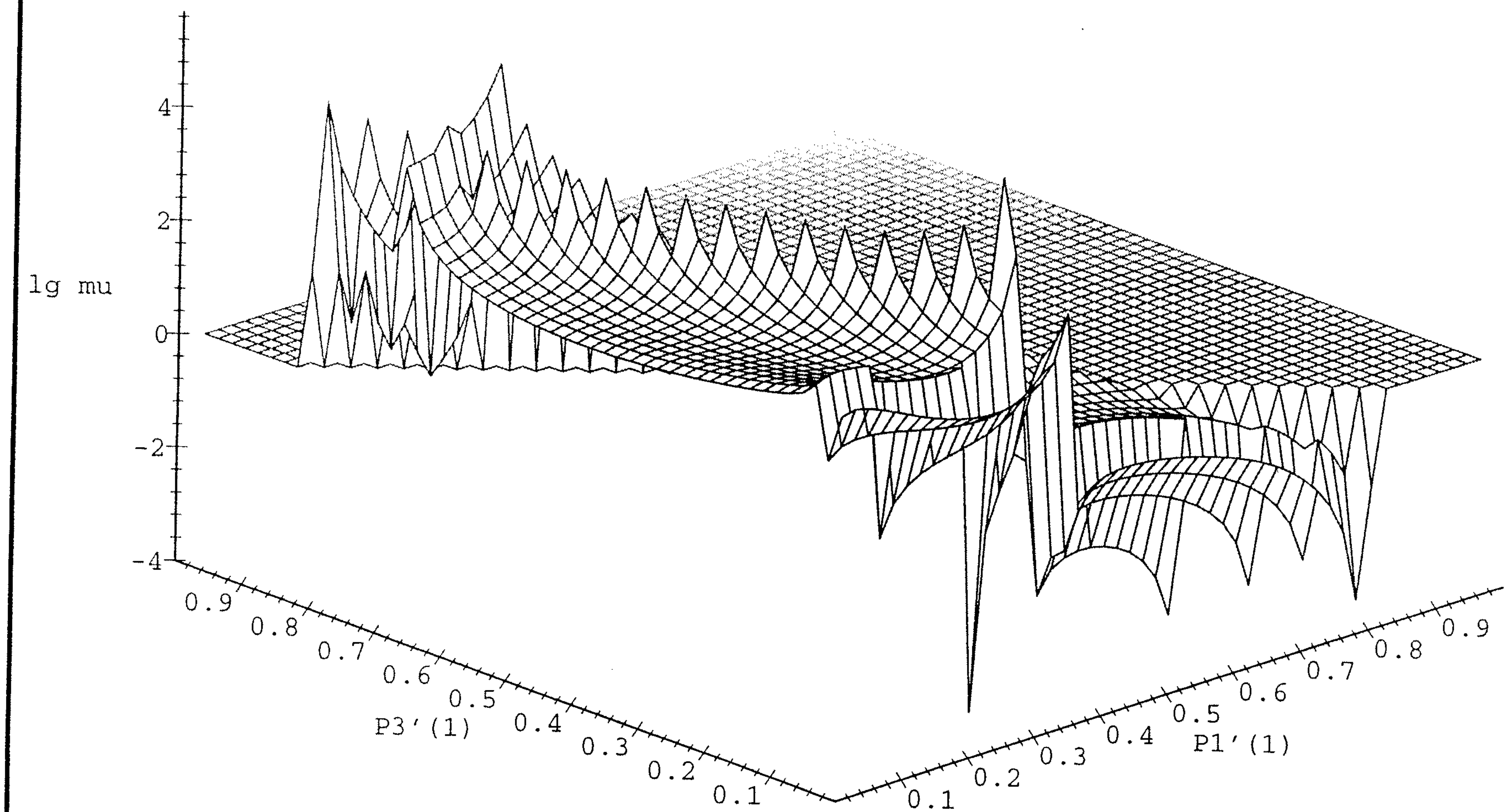
Plot 1.c-1.cT: E = {2,2,2}, T = {5,10,15}



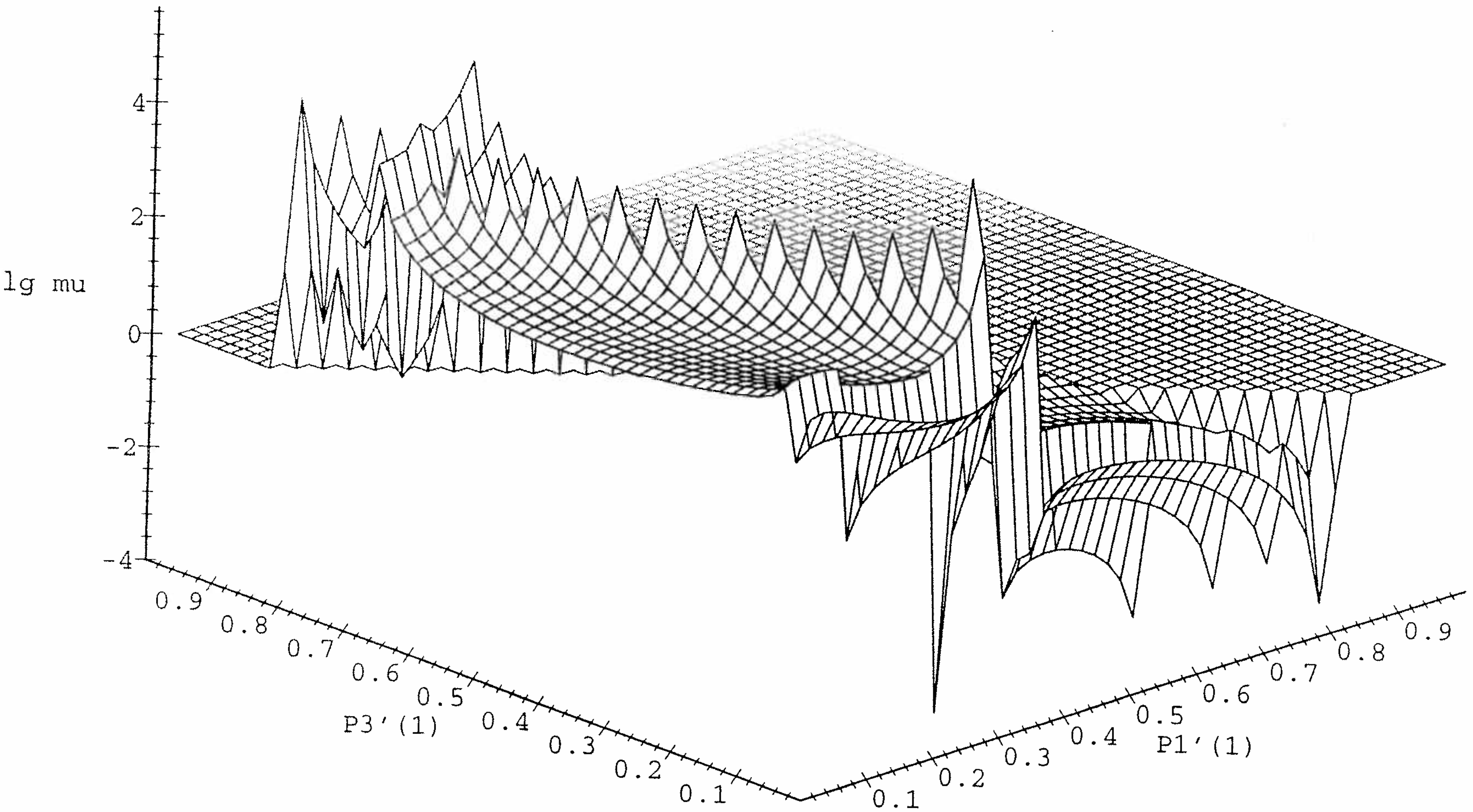
Plot 1a-1bT: E = {1,2,4}, T = {5,10,15}



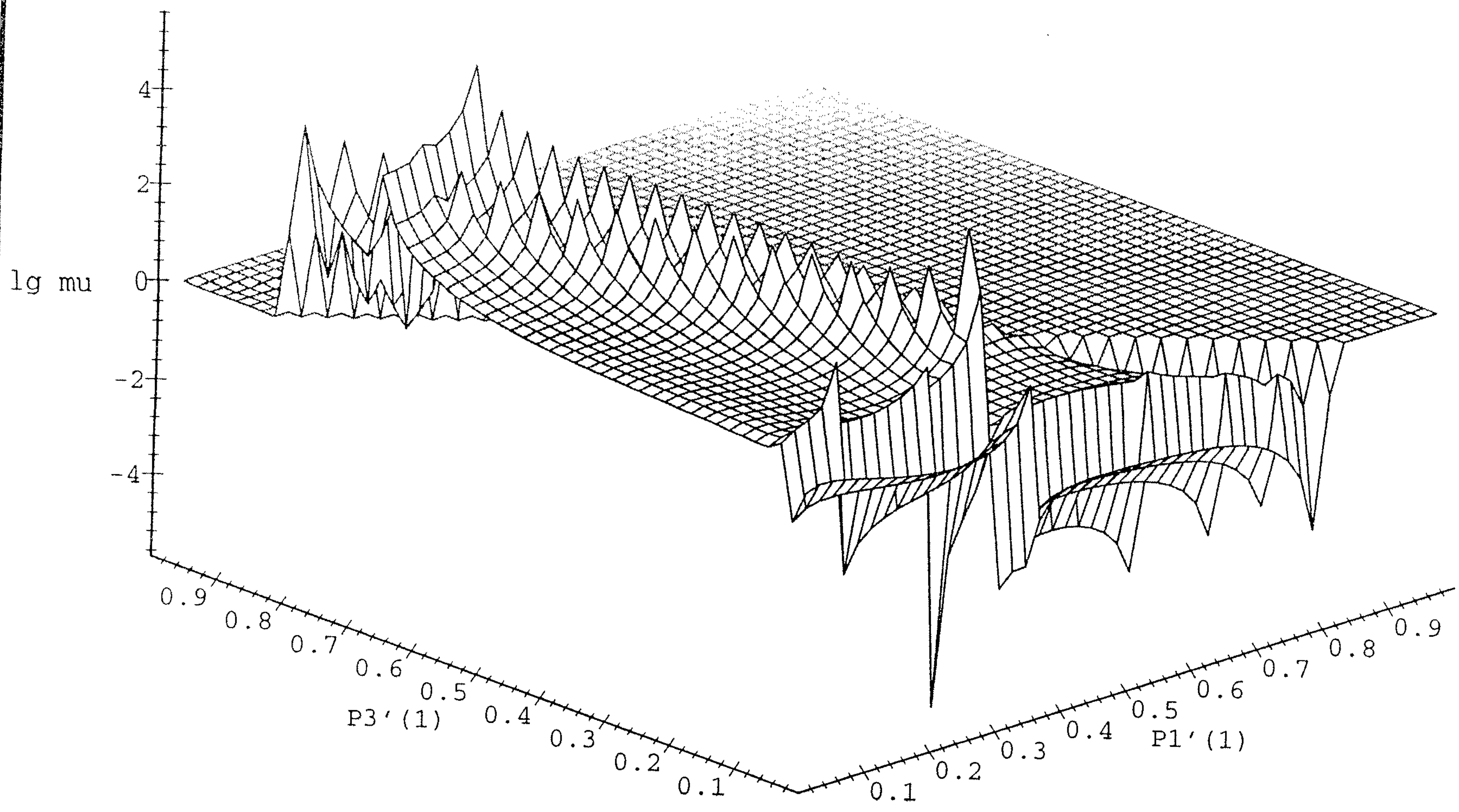
Plot 1a-1bT: E = {1,2,4}, T = {5,5,5}



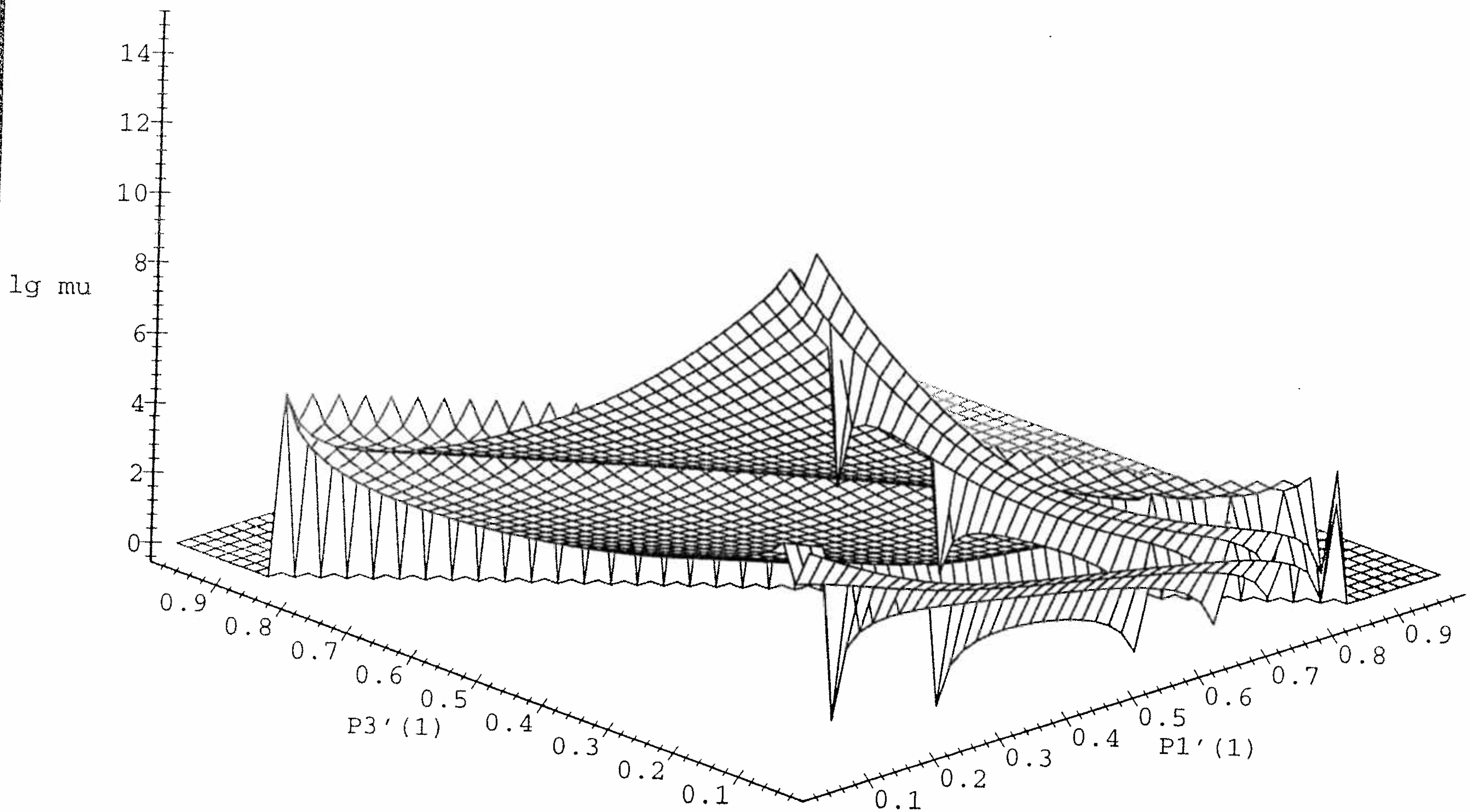
Plot 1a-1bT: E = {1,2,4}, T = {5,5,5}



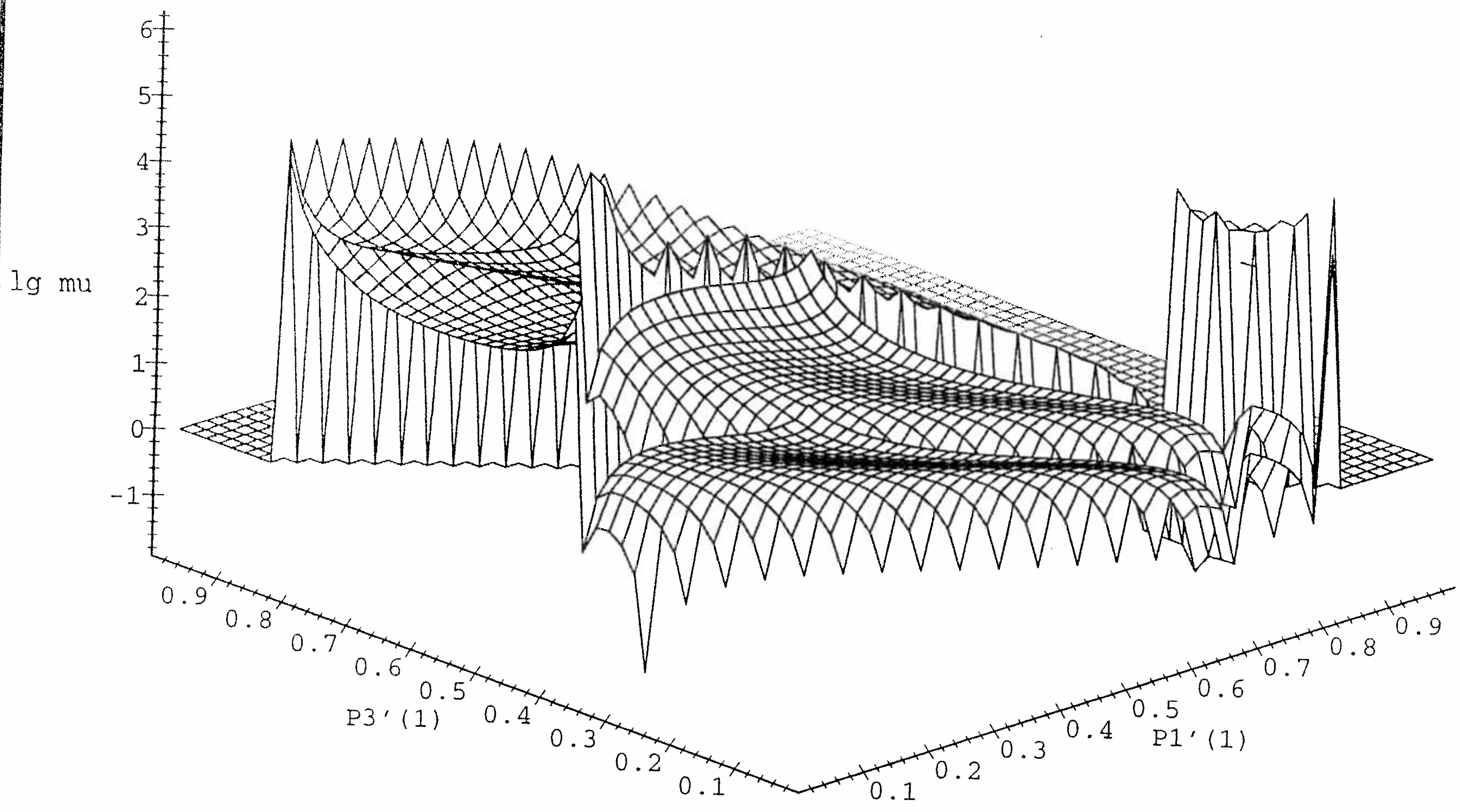
Plot 1a-1bT: E = {1,2,4}, T = {15,10,5}



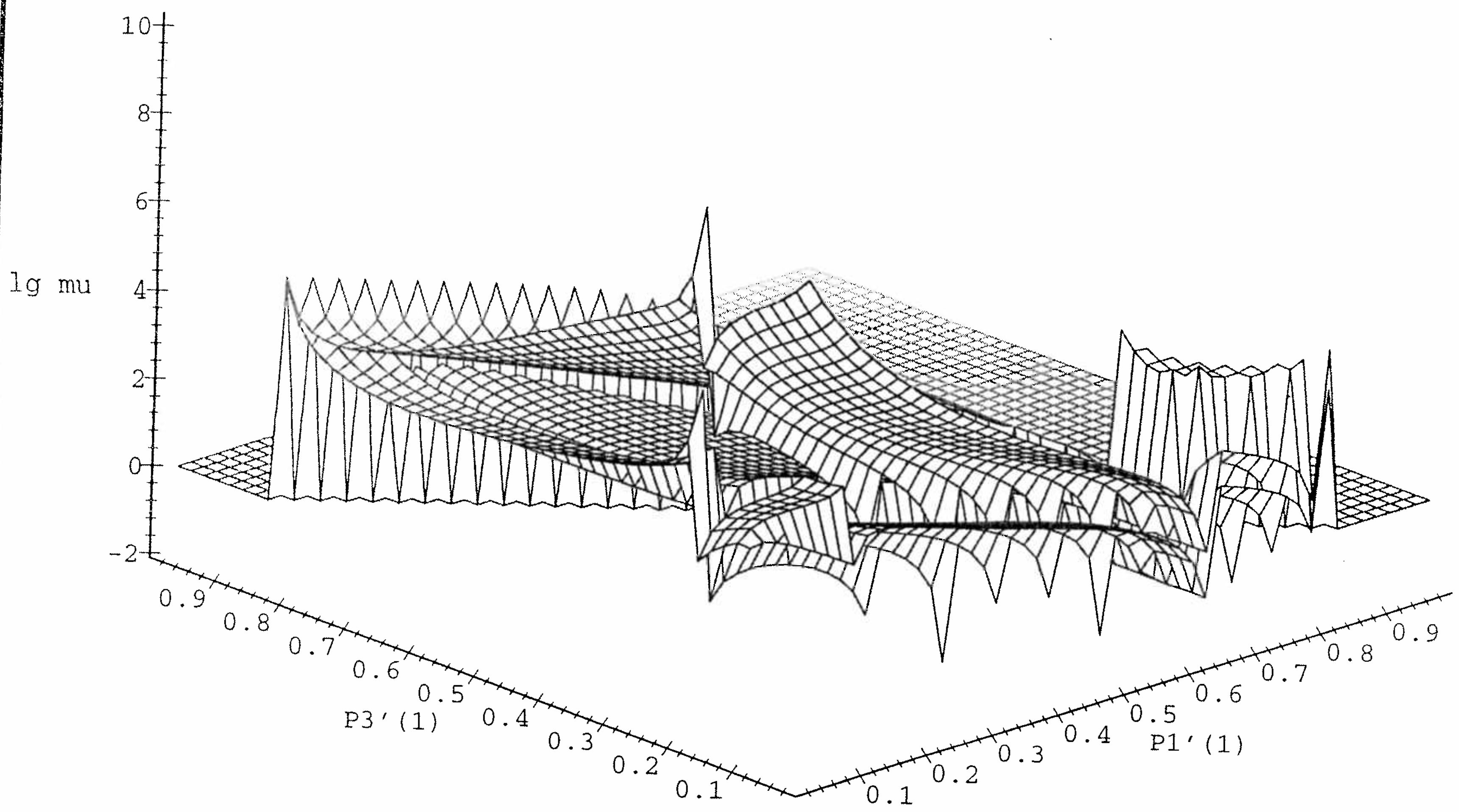
Plot **2**.a (10): E = {1,2,4}, T = {5,5,5}



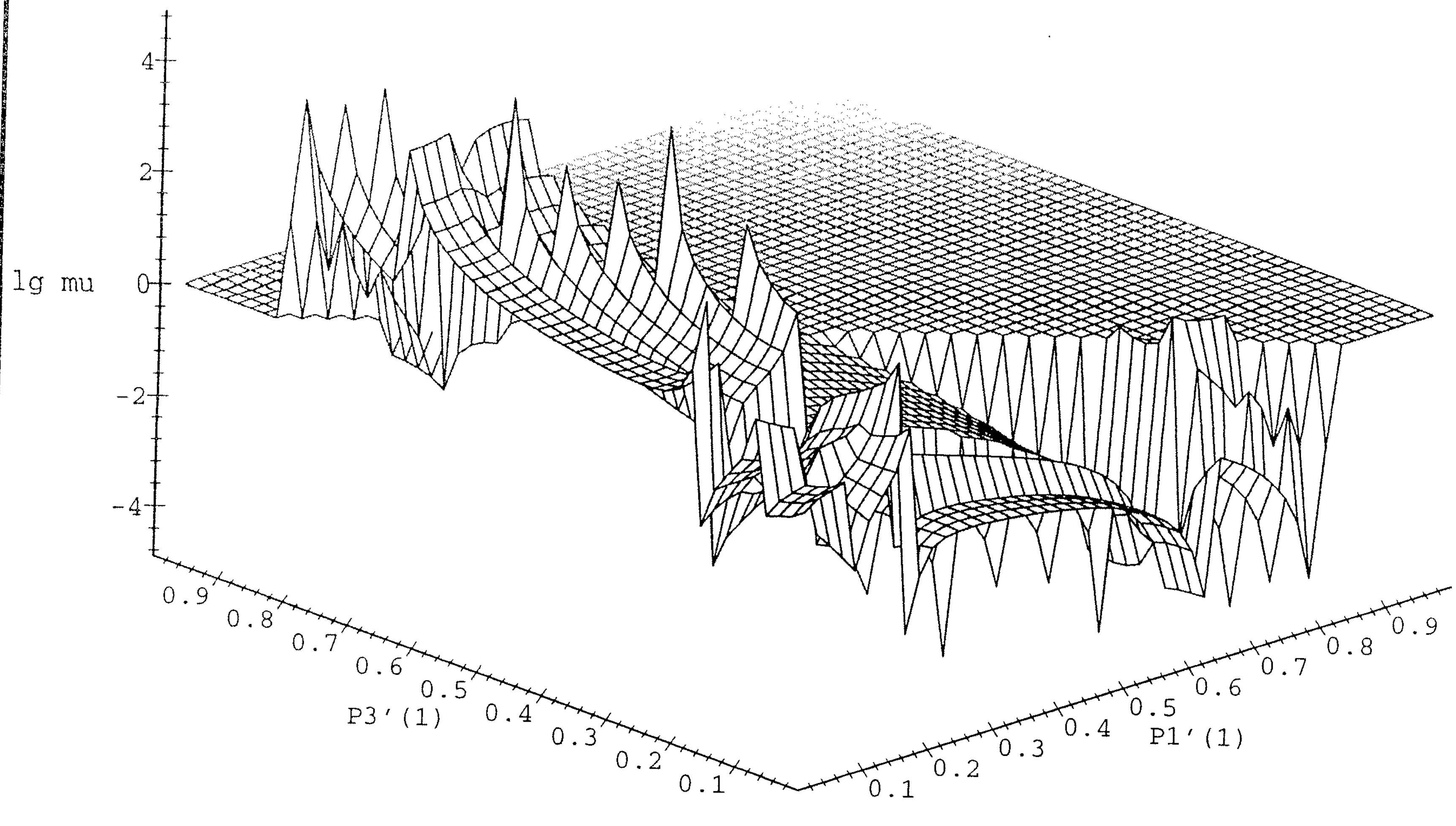
Plot 1.b (10): E = {4,2,1}, T = {5,5,5}



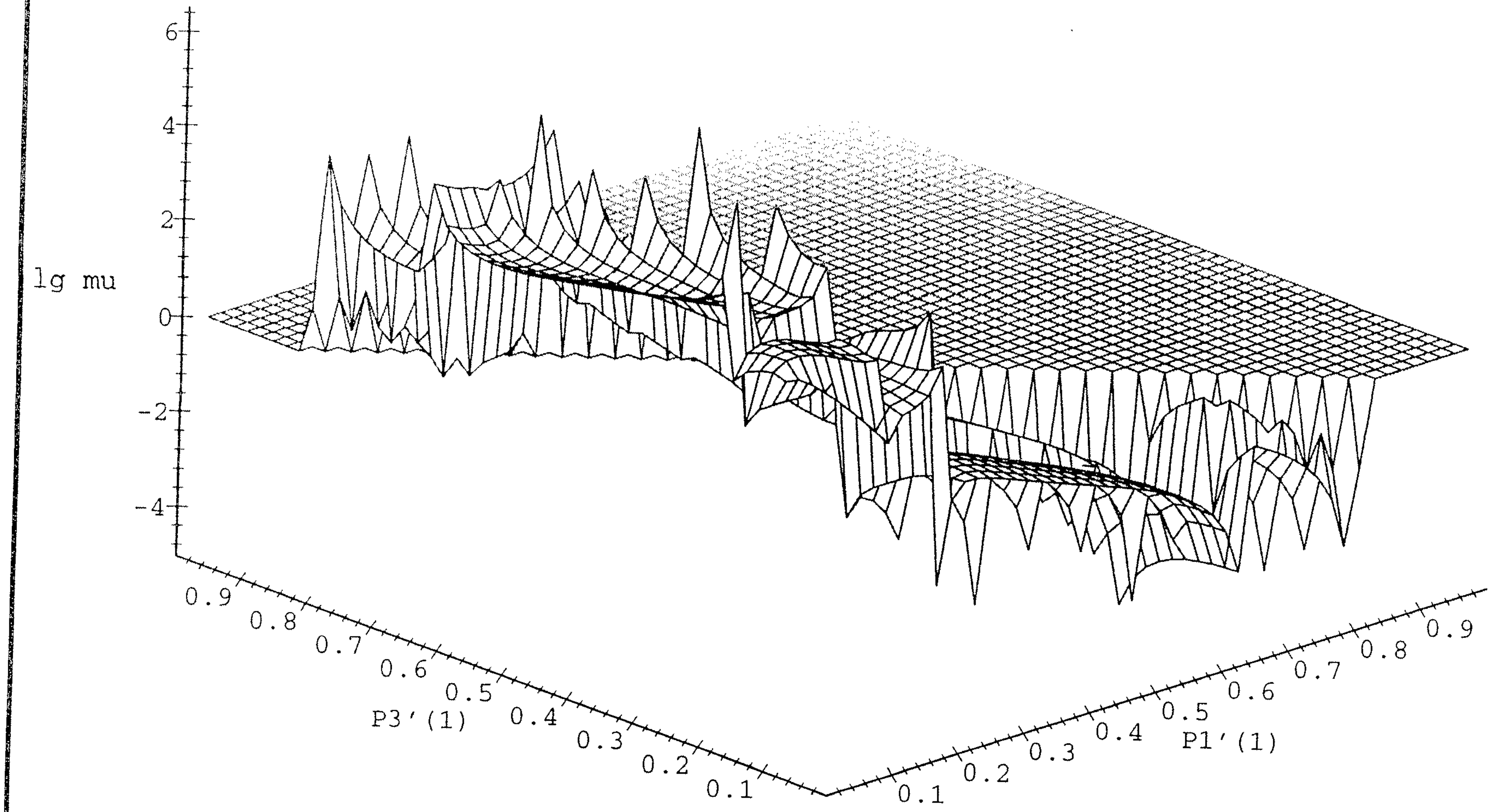
Plot 2.c (10): E = {2,2,2}, T = {15,15,15}



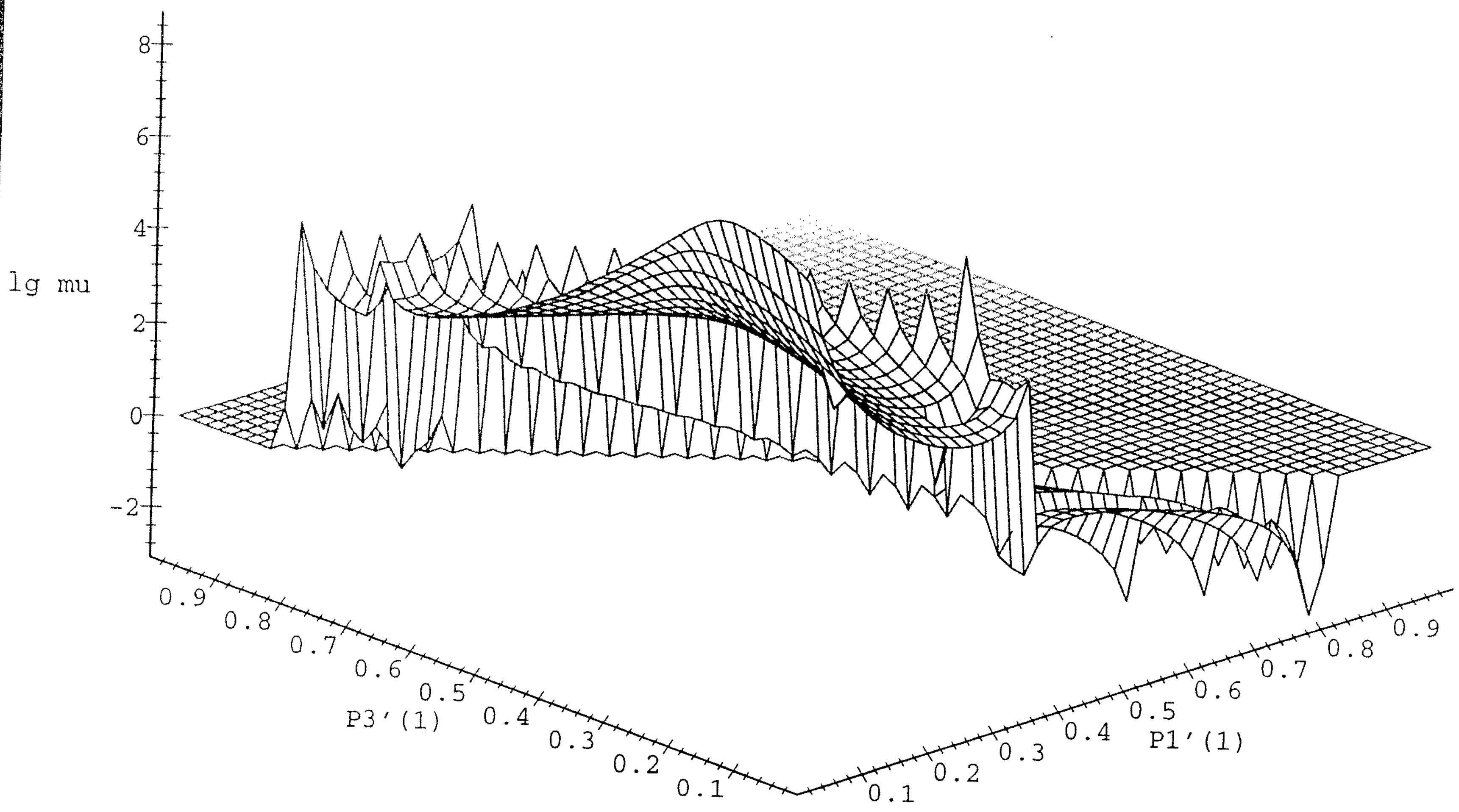
Plot 2.c-2.cT (10): E = {2,2,2}, T = {5,5,5}



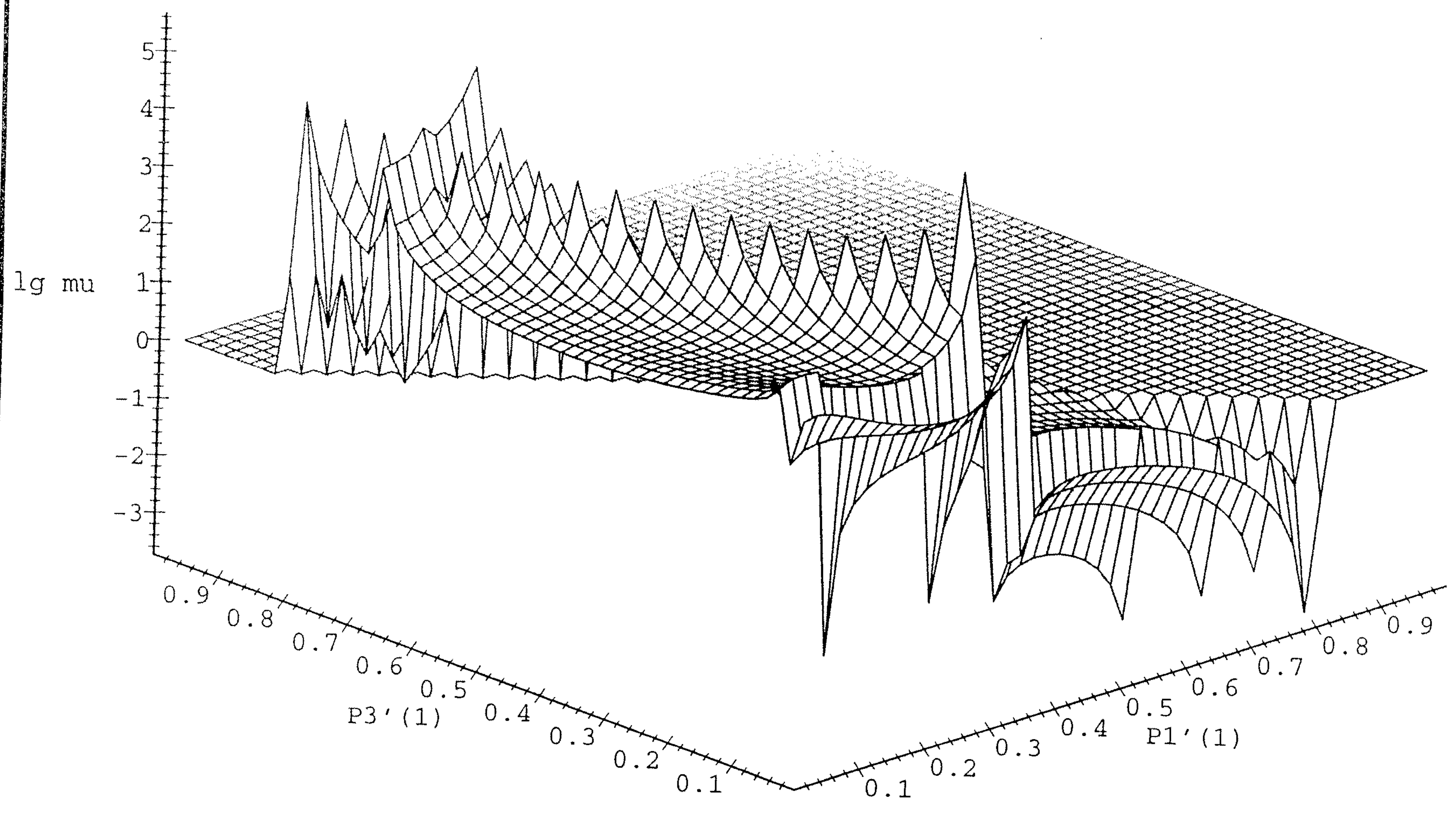
Plot **1**.c-**1**.cT (10): E = {2,2,2}, T = {5,10,15}



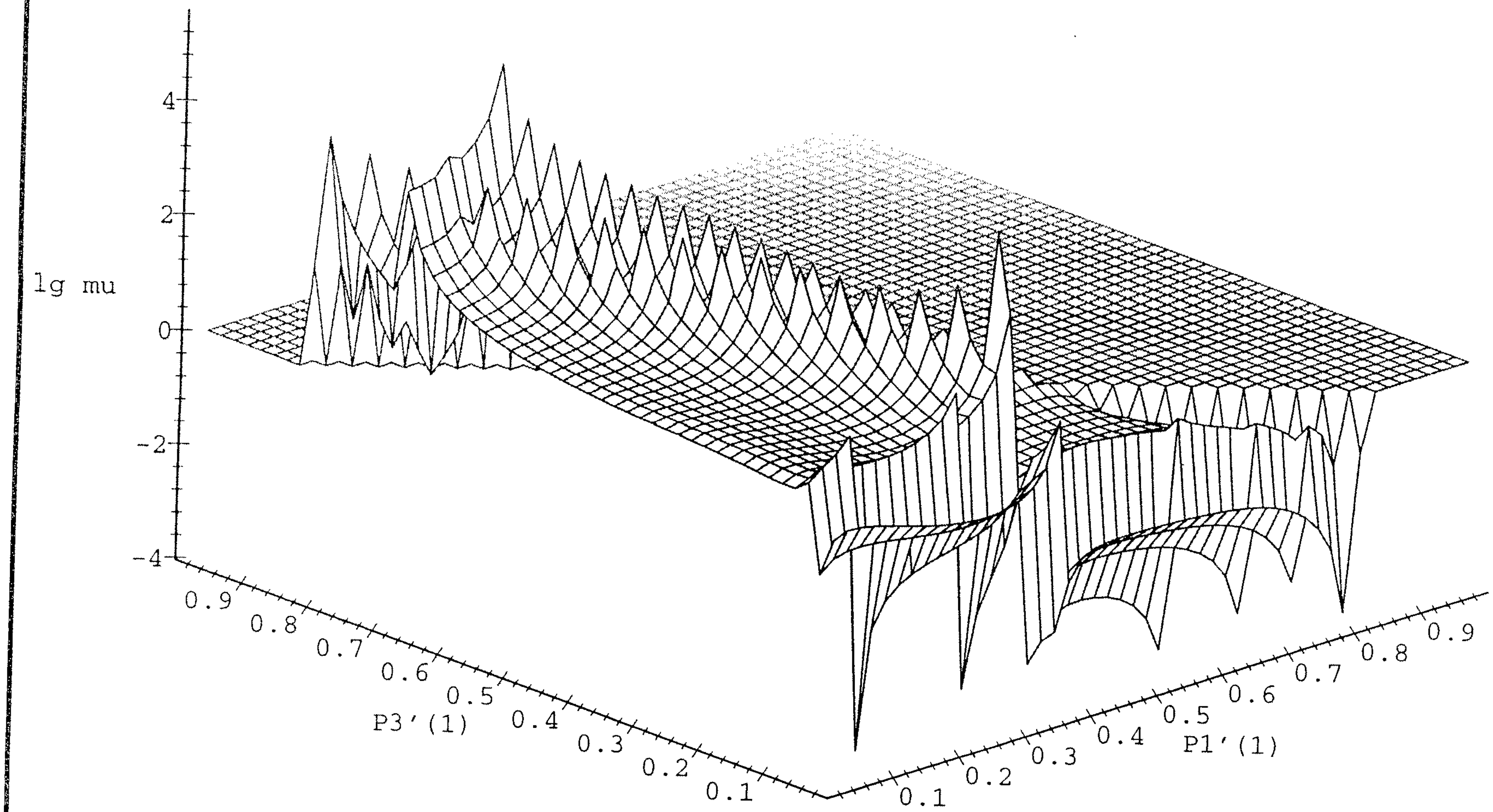
Plot 1.a-1.bT (10): E = {1,2,4}, T = {5,10,15}



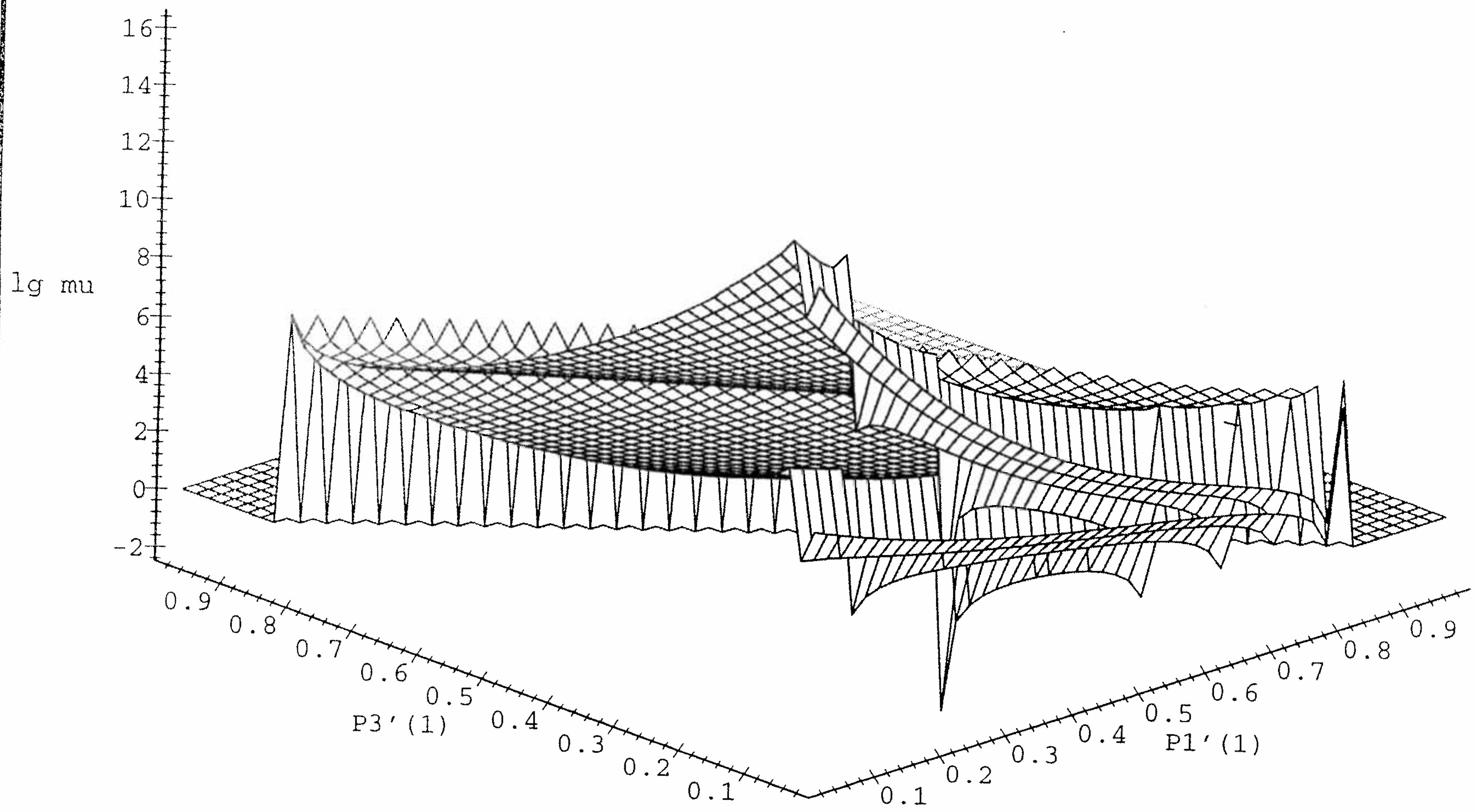
Plot 1.a-1.bT (10): E = {1,2,4}, T = {5,5,5}



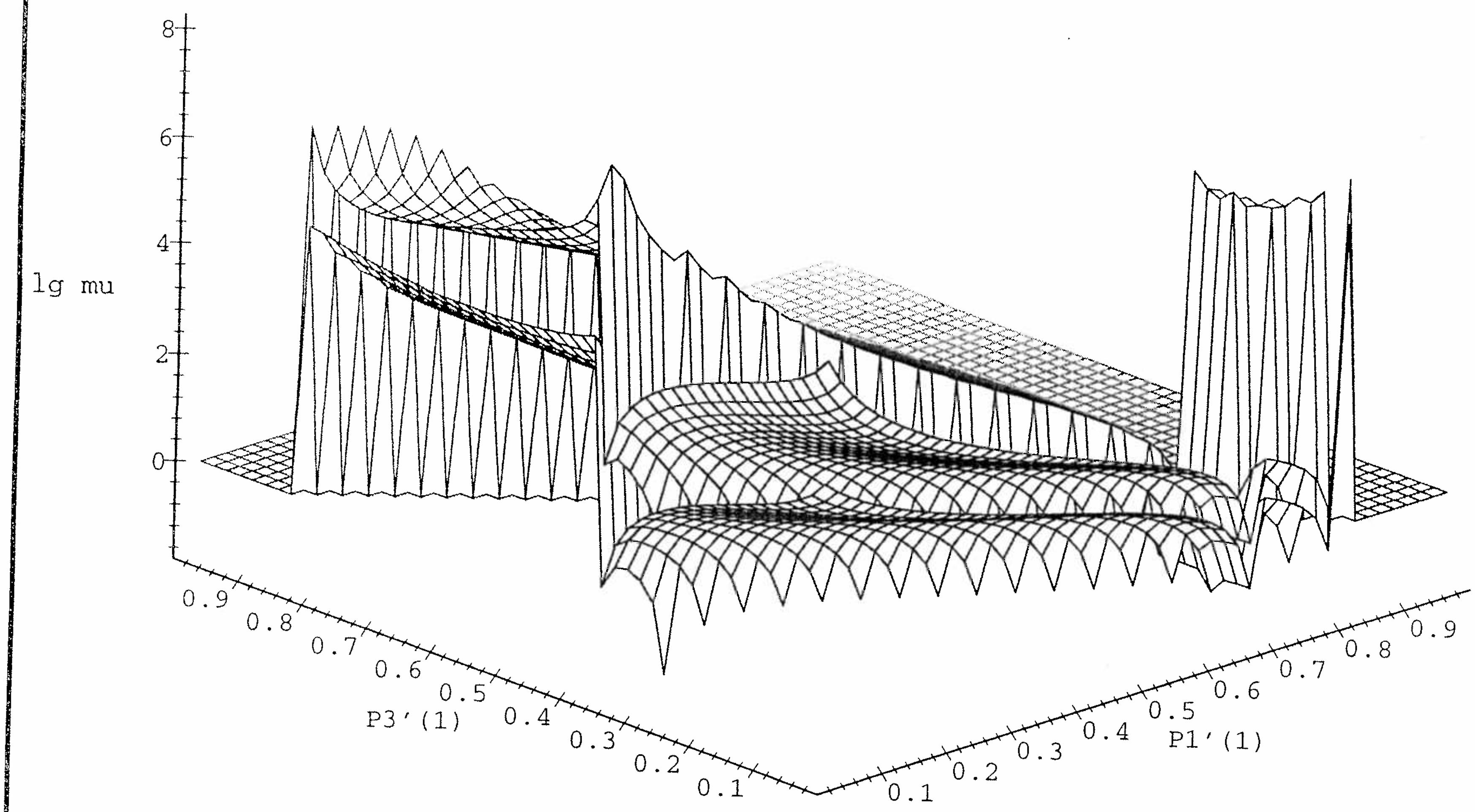
Plot **2**.a-**2**.bT (10): E = {1,2,4}, T = {15,10,5}



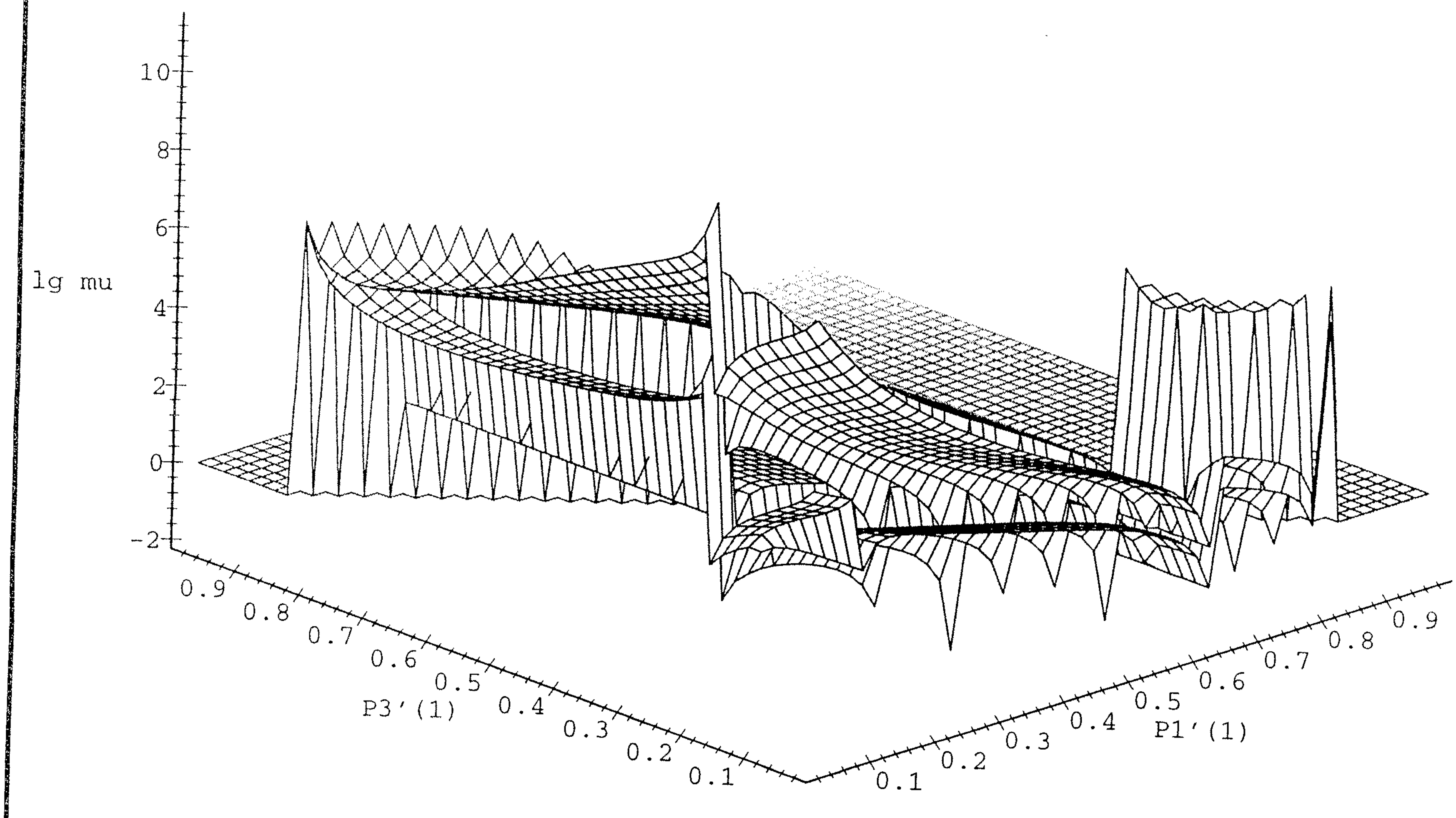
Plot 3.a (noD): E = {1,2,4}, T = {15,15,15}



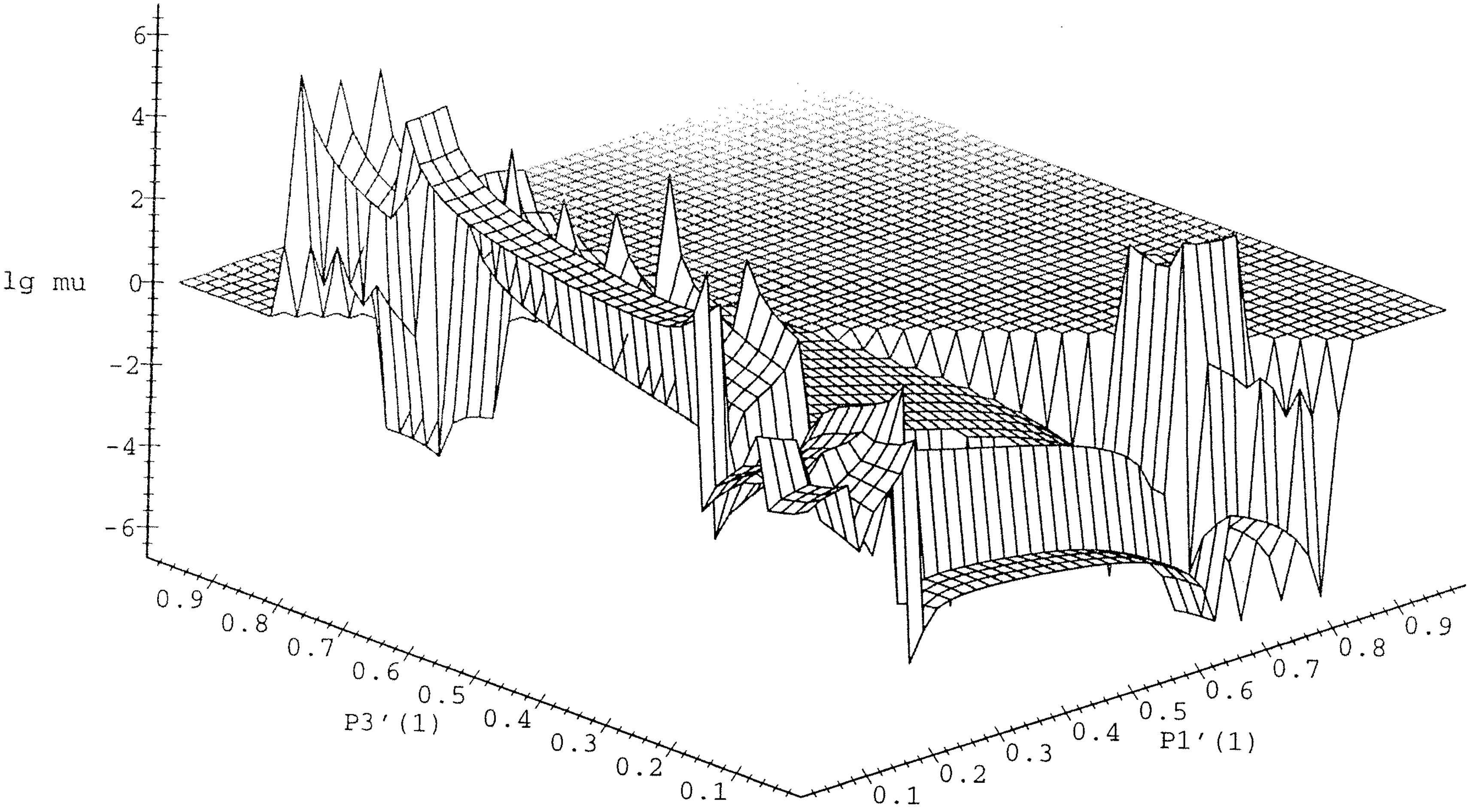
Plot 3b (noD): E = {4,2,1}, T = {15,15,15}



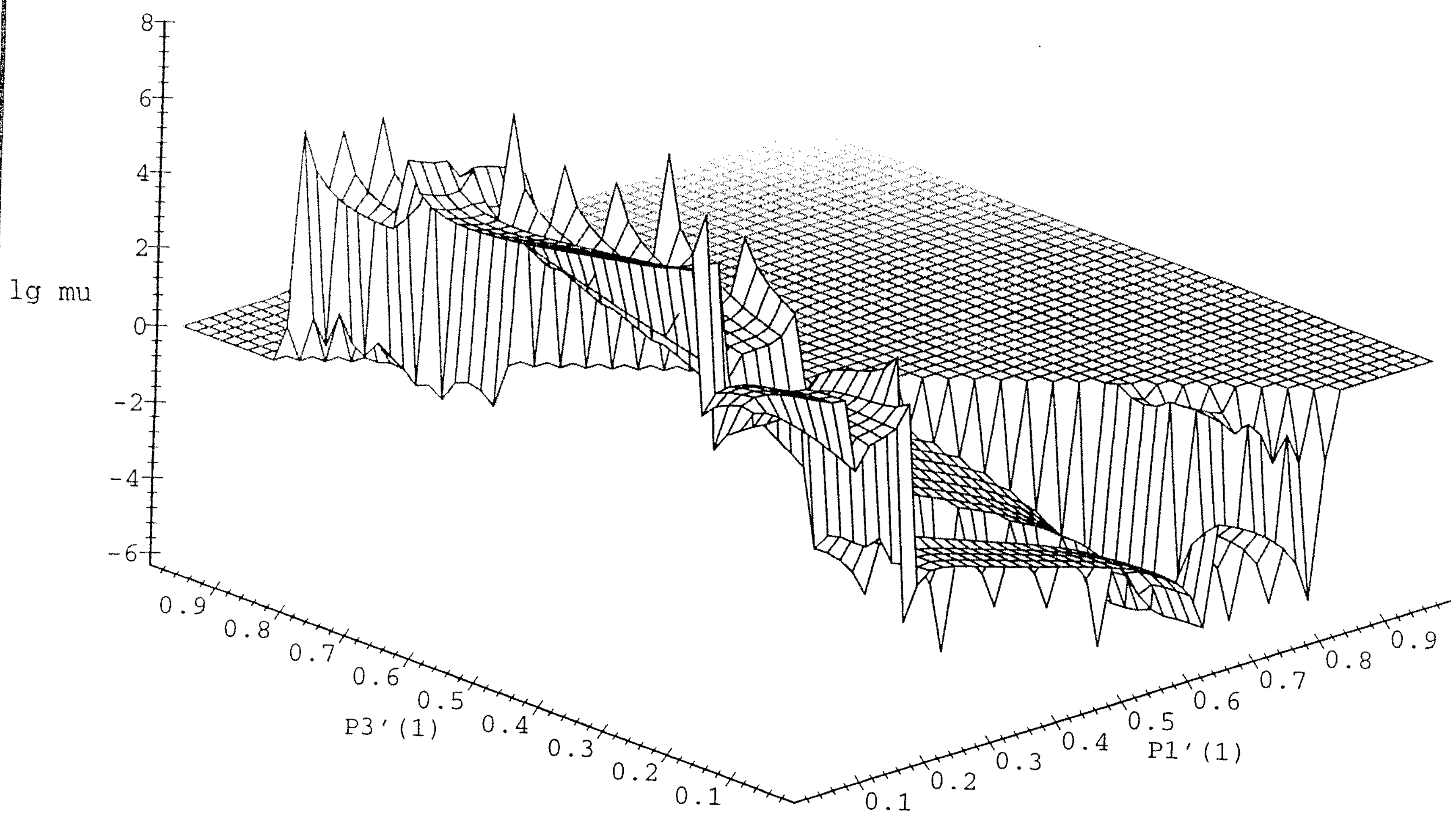
Plot 3.c (noD): E = {2,2,2}, T = {5,5,5}



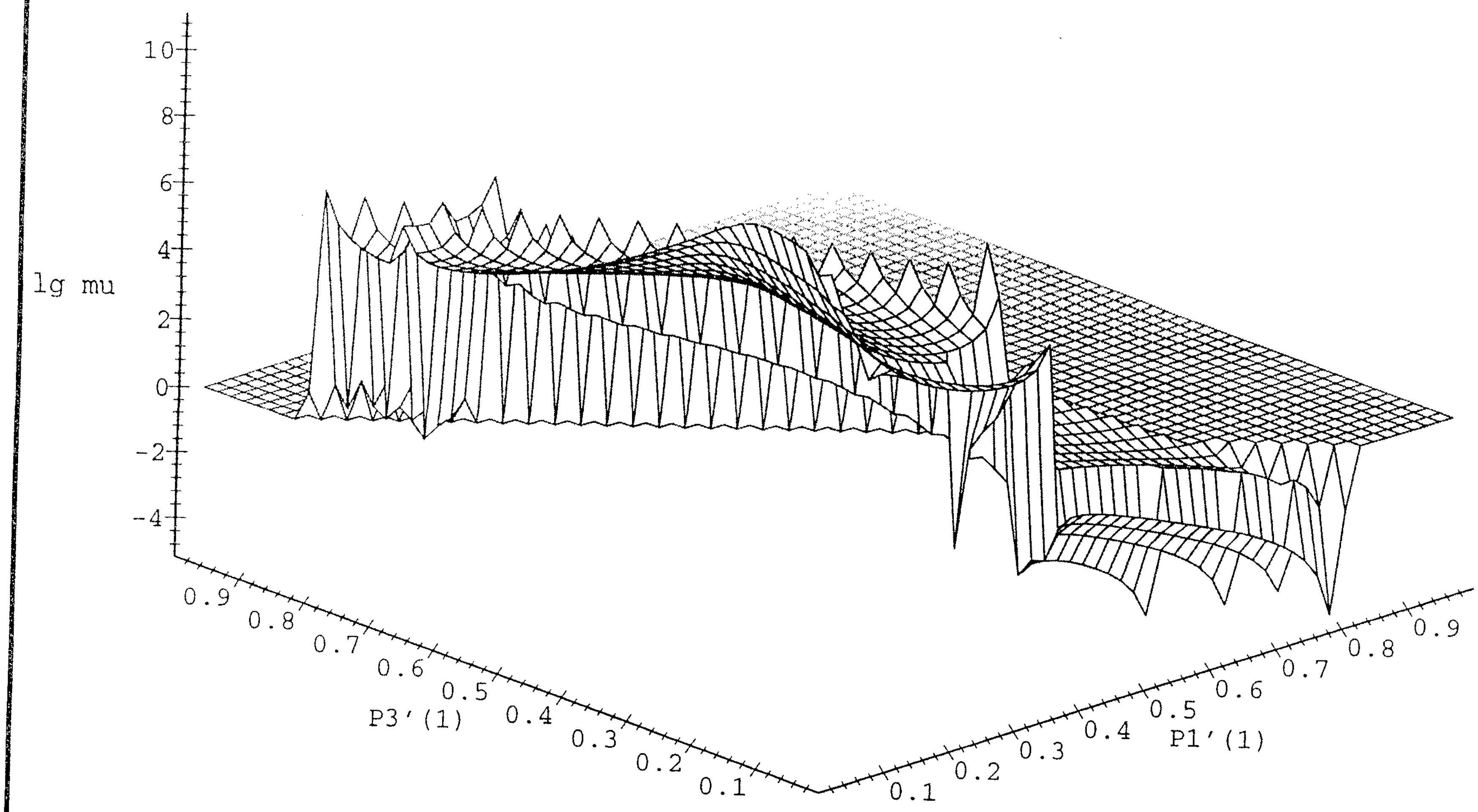
Plot 3.c-3.cT (noD): E = {2,2,2}, T = {5,5,5}



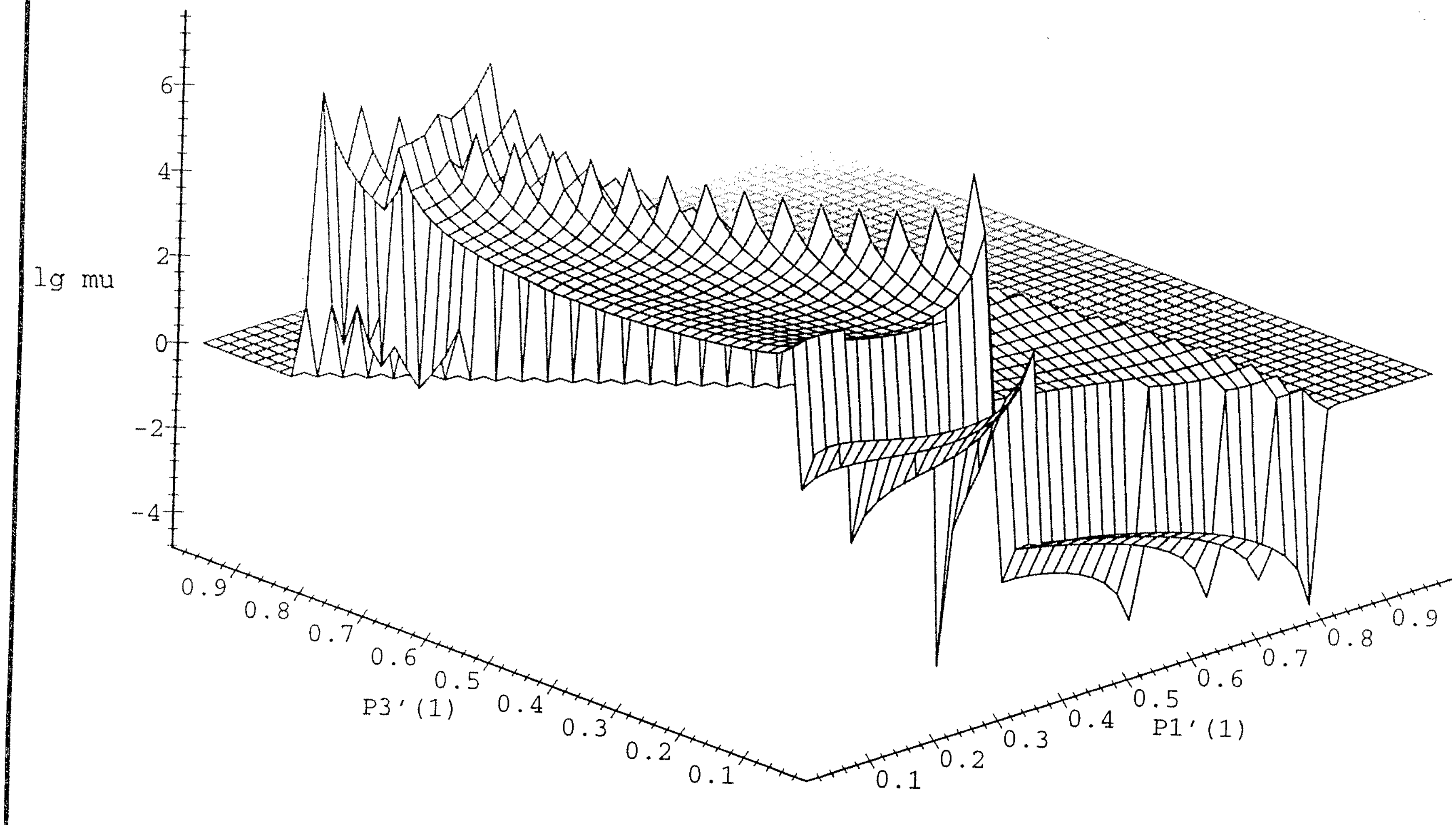
Plot **3**.c-**3**.cT (noD): E = {2,2,2}, T = {5,10,15}



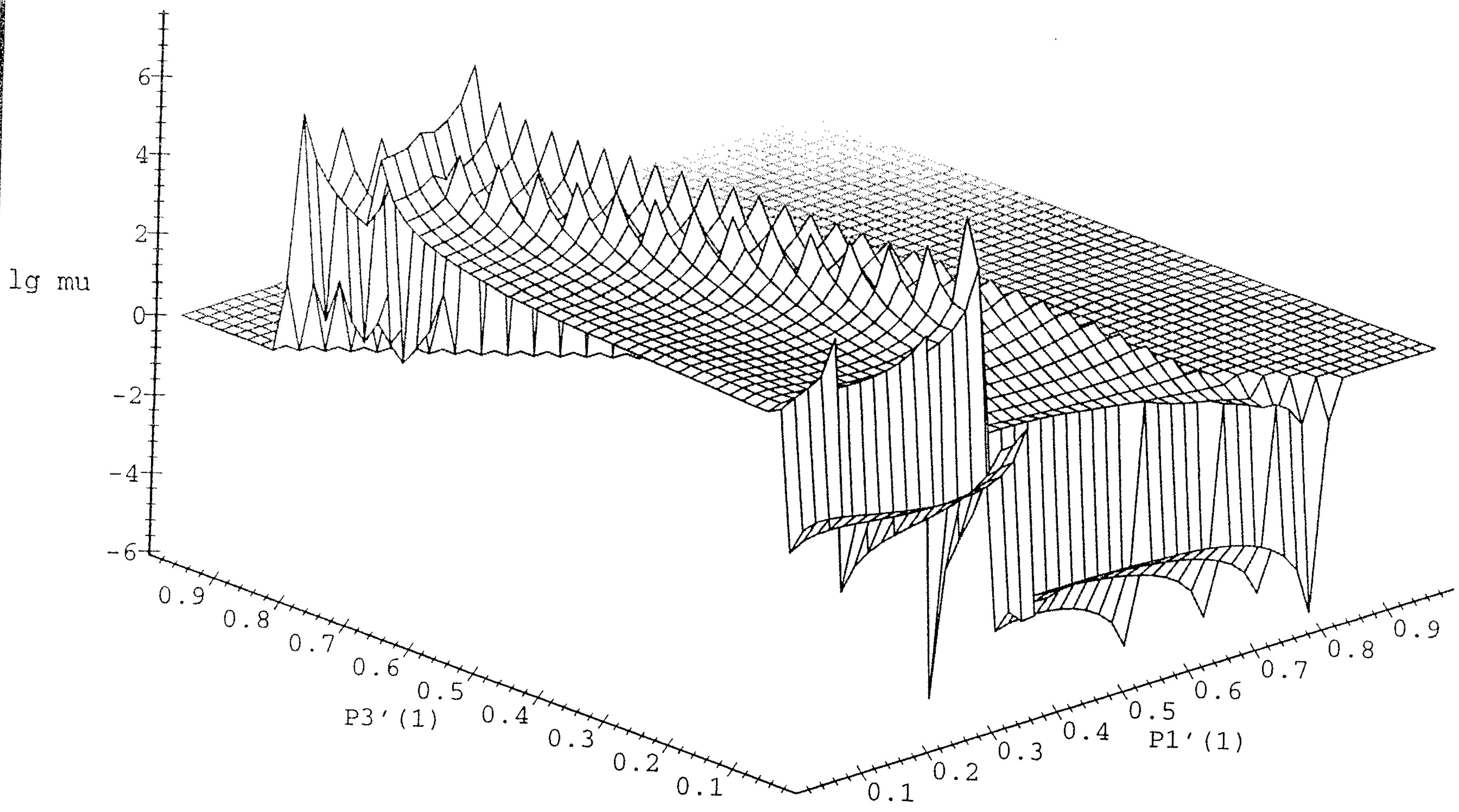
Plot 3.a-3.bT (noD): E = {1,2,4}, T = {5,10,15}



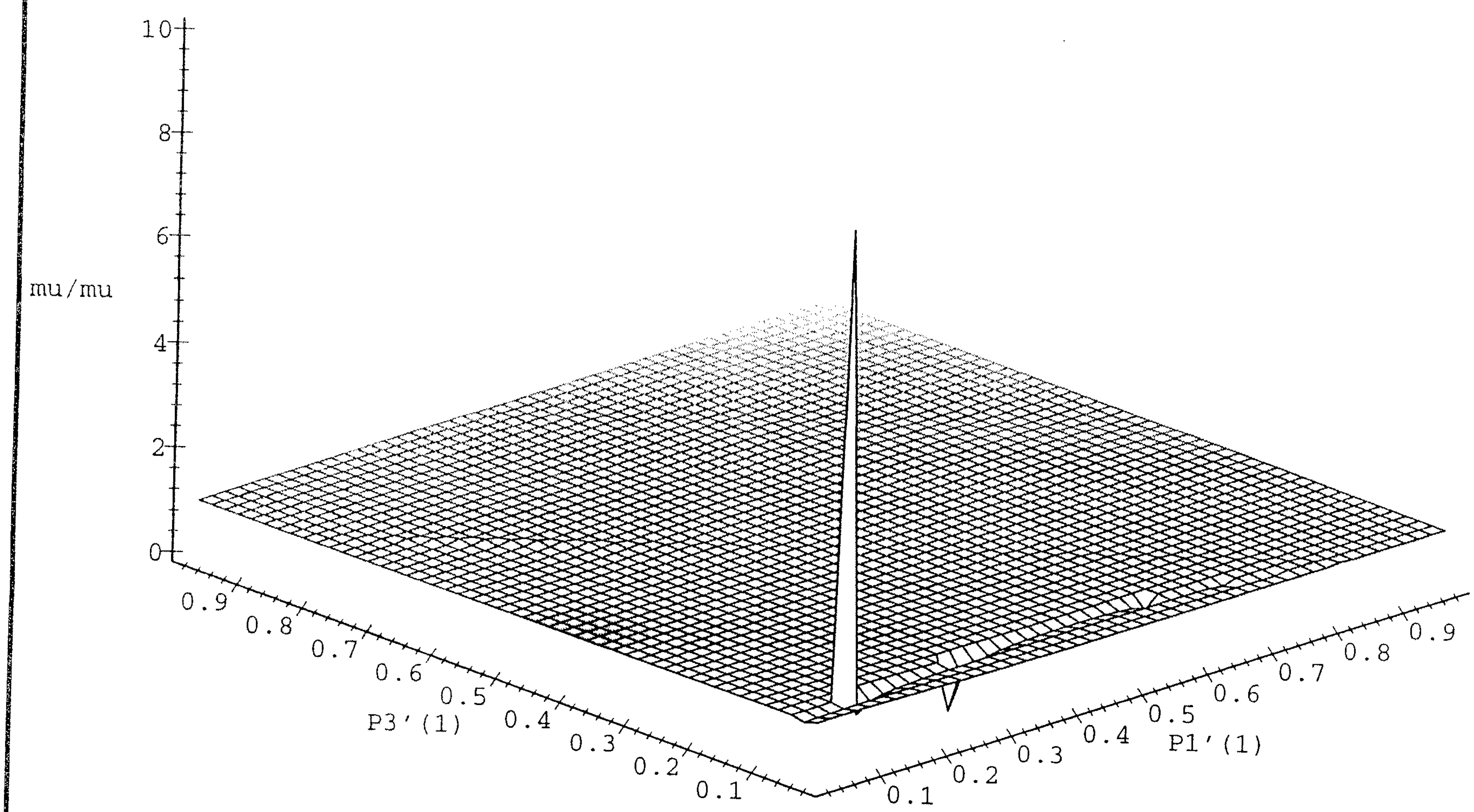
Plot 3.a-3.bT (noD): E = {1,2,4}, T = {5,5,5}



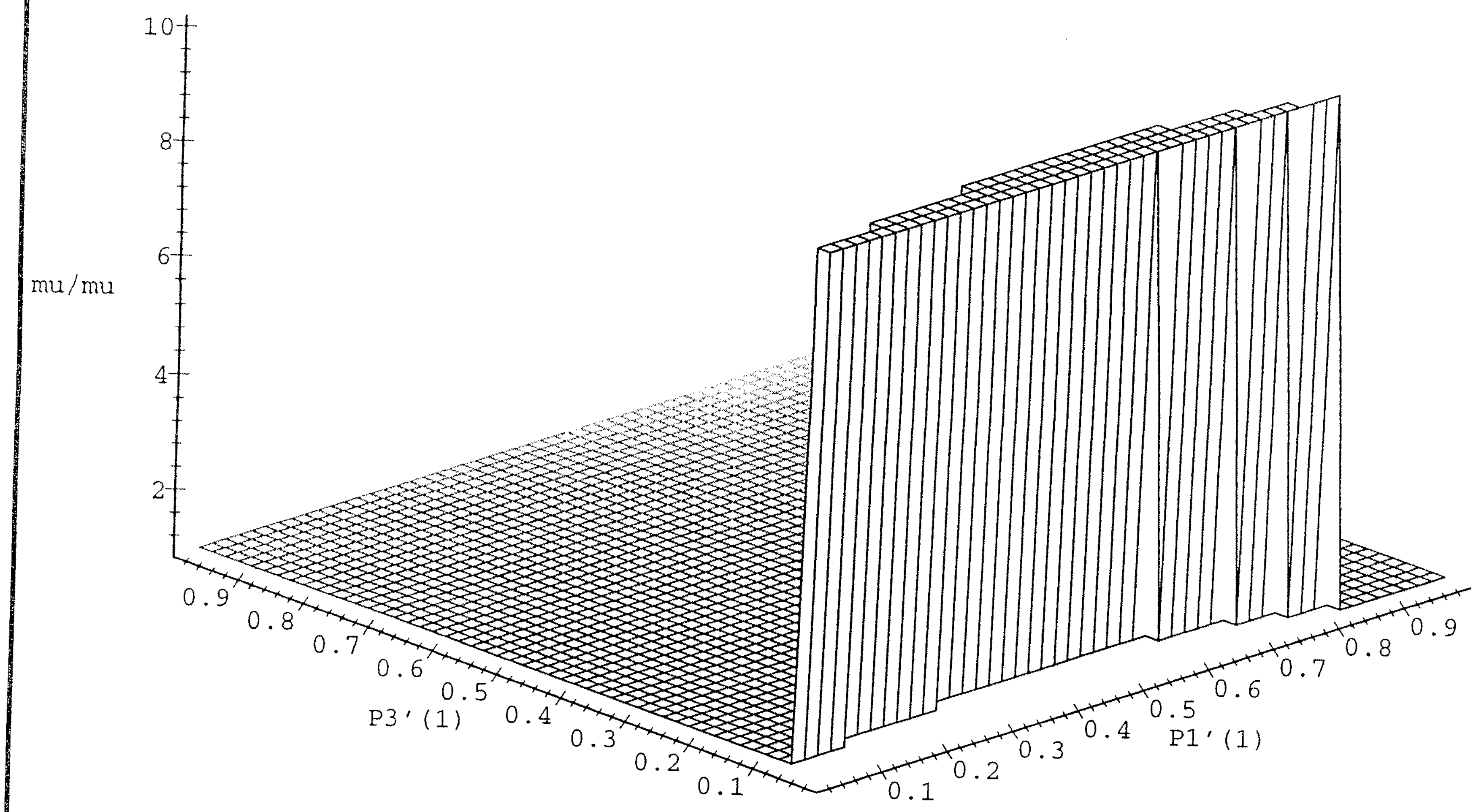
Plot 3.a-3.bT (noD): E = {1,2,4}, T = {15,10,5}



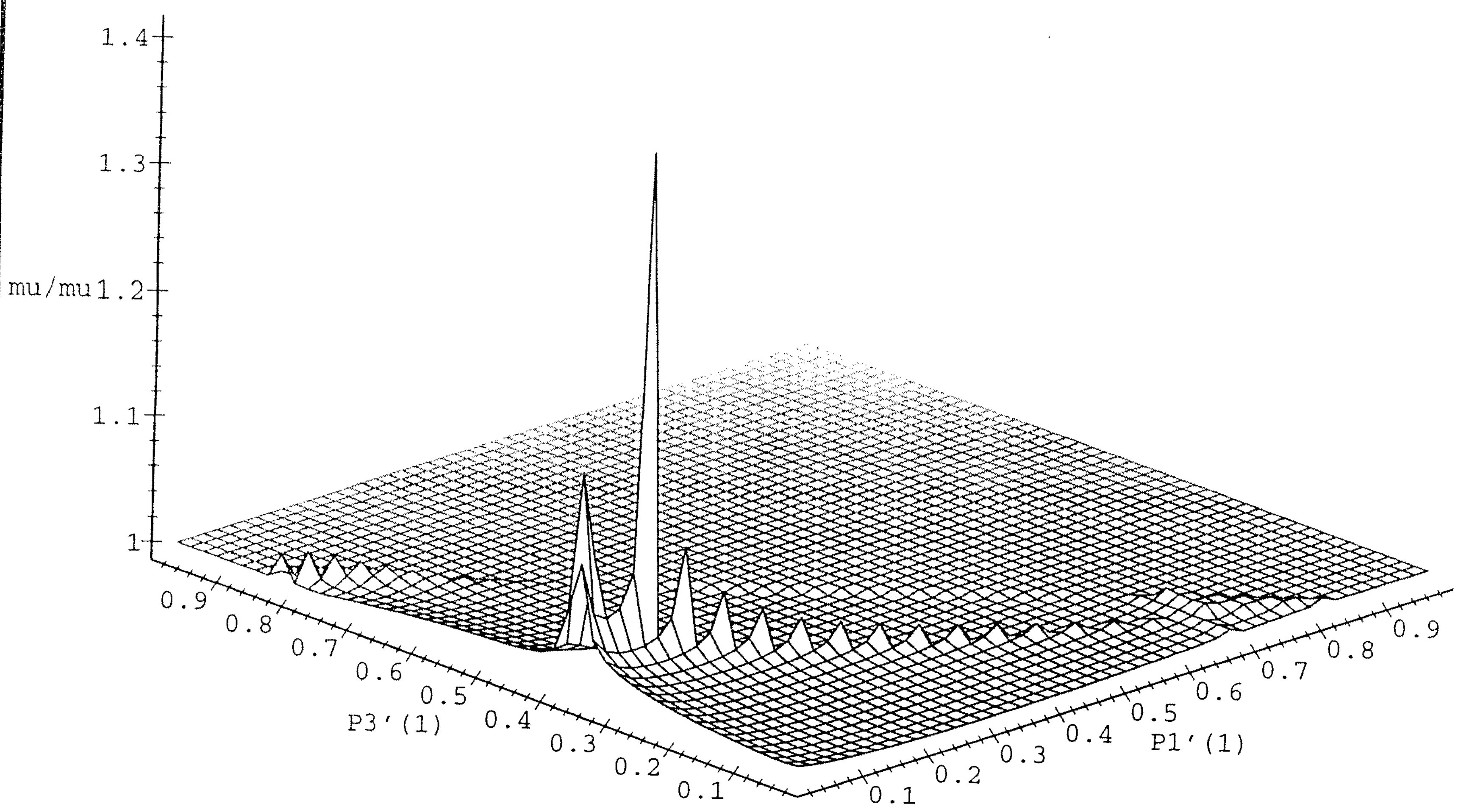
Plot 1a-2a'(10): l1=1000, l2=2000, l3=4000 (T1=5000, T2=10000, T3=20000)



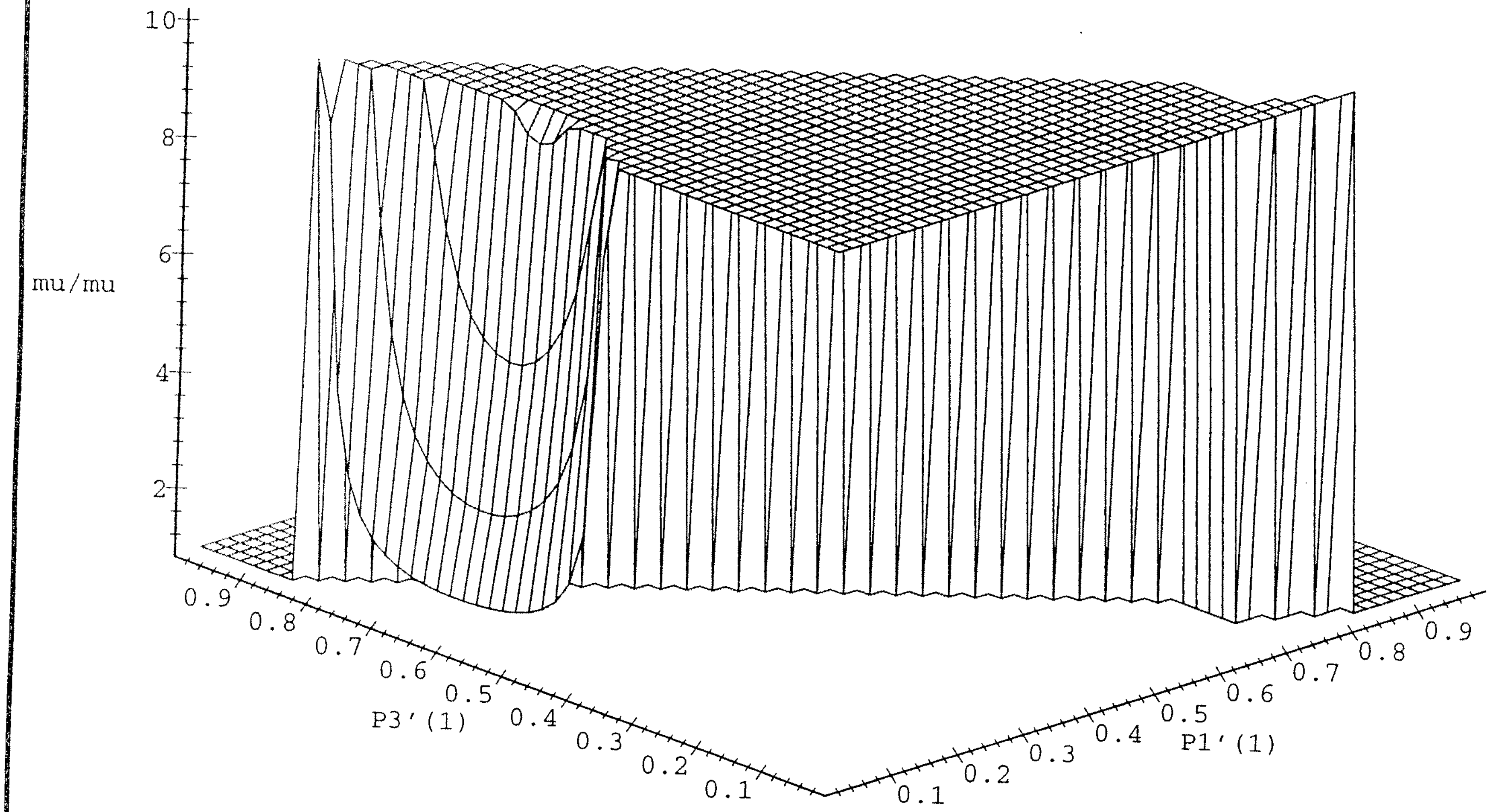
Plot 1a-3a(noD): l1=1000, l2=2000, l3=4000 (T1=5000, T2=10000, T3=20000)



Plot 1b-2b'(10): l1=4000, l2=2000, l3=1000 (T1=20000, T2=10000, T3=5000)

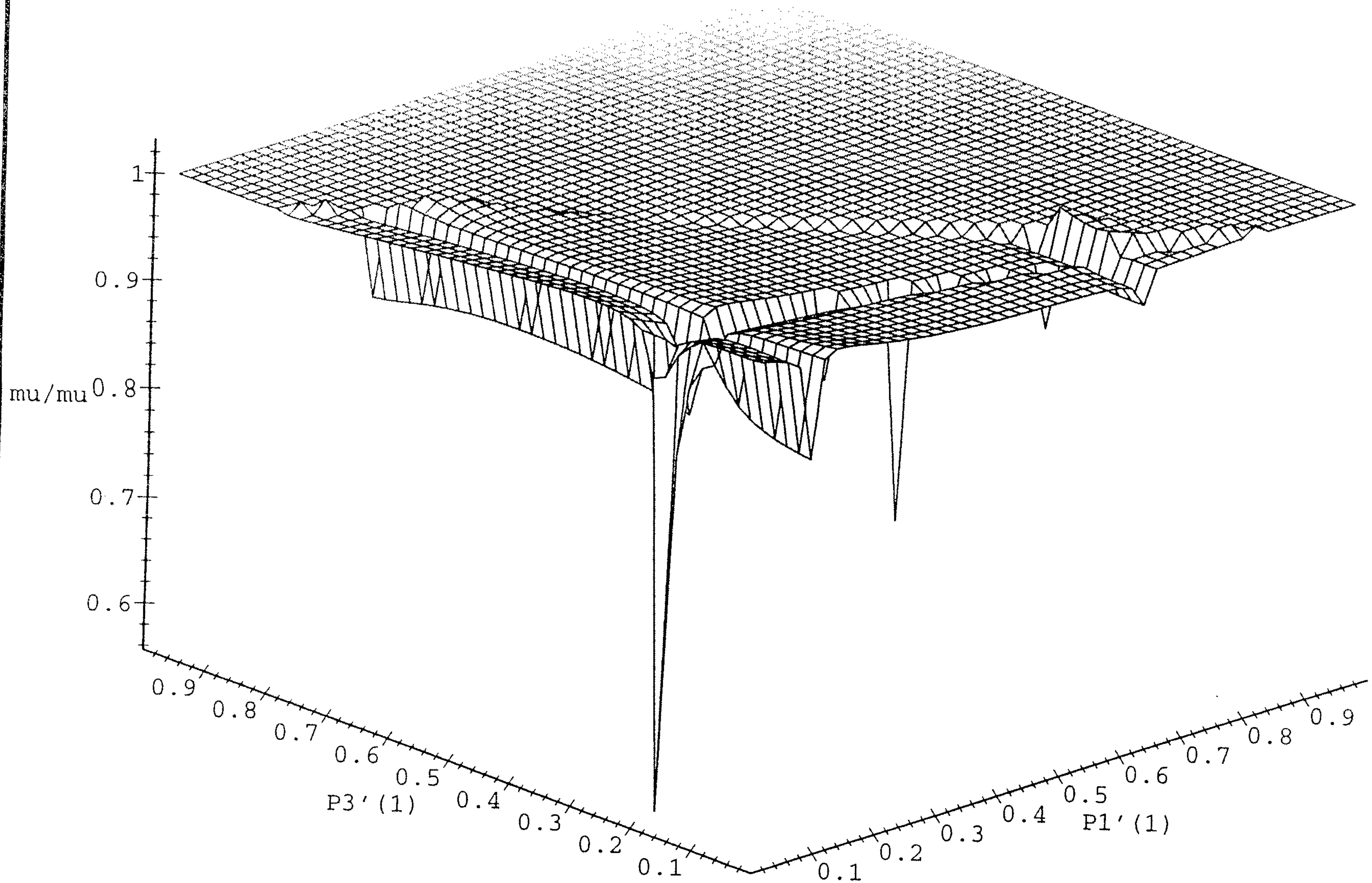


Plot 1b-3b(noD): l1=4000, l2=2000, l3=1000 (T1=20000, T2=10000, T3=5000)



Plot 2-2'(10): l1=2000, l2=2000, l3=2000 (T1=10000, T2=10000, T3=10000)

1.c - 2.c



Plot 2-2(noD): I1=2000, I2=2000, I3=2000 (T1=10000, T2=10000, T3=10000)

1.c - 3.c