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On Nonpreemptive LCFS Scheduling with Deadlines*

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We investigate some real time behaviour of a (discrete time) single server system with nonpreemptive LCFS task scheduling. The main results deal with the probability distribution of a random variable SRD(T), which describes the time the system operates without any violation of a fixed task service time deadline T. A tree approach, similar to those already used for the derivation of the same quantities for other scheduling disciplines (e.g., FCFS) is suitable here again, establishing the power of such techniques once more. Relying on a simple general probability model, asymptotic formulas concerning all moments of SRD(T) are determined; for example, the expectation of SRD(T) is proved to grow exponentially in T, i.e., $E[\mathrm{SRD}(T)] \sim CT^{3/2}\rho^T$ for some $\rho > 1$. Our computations rely on a multivariate (asymptotic) coefficient extraction technique which we call asymptotic separation.

1. BASICS

In this paper we shall study some aspects concerning the real time behavior of a discrete time single server system with nonpreemptive LCFS task scheduling. Instead of using queueing theory, we apply a special tree approach already used for the derivation of similar results in the case of FCFS and preemptive LCFS scheduling; see [2, 8]. Both papers contain a very detailed introduction to the model, as well.

The outline of the paper is as follows: After a short description of the underlying model and some questions of interest, we provide the tree approach suitable for the combinatorial and asymptotic computations in Section 3 and 4. Section 5 is devoted to our final results. Finally, some conclusions are appended in Section 6.

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We consider a system containing a task scheduler, a task list, and a single server. Tasks arriving at the system are taken by the scheduler and placed into the task list according to the scheduling strategy. The server always executes the task at the head of the list; thus scheduling is done by rearranging the task list. A dummy task will be generated by the scheduler if the list becomes empty. If the server executes a dummy task the system is called *idle*, otherwise *busy*.

Rearranging the task list is assumed to occur at discrete points on the time axis only, without any overhead. The (constant) time interval between two such points is called a *cycle*. Due to this assumption we are able to model tasks formed by indivisible (atomic) actions with duration of 1 cycle. The *task execution time* of a task is the number of cycles necessary for processing the task to completion if it might occupy the server exclusively. A "regular" task may have an arbitrary task execution time; a dummy task as mentioned above is supposed to consist of a single no-operation action (one cycle). The *service time* of a task is the time (measured in cycles) from the beginning of the cycle in which the corresponding task arrives at the system to the end of the last cycle of that task.

Obviously, the time axis is covered by *busy periods*, which are supposed to include the initial idle cycle, too. Note that this definition implies the correspondence between an idle cycle and a busy period with duration of 1 cycle. A sequence of busy periods without any violation of any task's service deadline, followed by a busy period containing at least one deadline violation, is called a *run*; the sequence without the last (violating) busy period is referred to as a *successful run*.

In order to investigate real time performance, we shall study the *successful run duration* SRD(T), which is the time interval from the beginning of the initial idle cycle to the beginning of the (idle) cycle initiating the busy period containing the first violation of a task's deadline T.

We assume an arrival process which provides an arbitrarily distributed number of task arrivals within a cycle, independent of the arrivals in the preceding cycles, and independent of the arbitrary distributed task execution times, as well.

The probability generating function (PGF) for the number of task arrivals during a cycle is denoted by

$$A(z) = \sum_{k \geq 0} a_k z^k \tag{1.1}$$

and should meet the constraint $a_0 = A(0) > 0$; i.e., the probability of no arrivals during a cycle should be greater than zero. This ensures the existence of idle cycles.

The PGF of the task execution time (measured in cycles) is denoted by

$$L(z) = \sum_{k \geq 0} l_k z^k \tag{1.2}$$

with the additional assumption $L(0) = 0$; i.e., the task execution time should be greater than or equal to one cycle. Note that we assume an a priori knowledge of the task execution time at the time the task arrives.

For technical reasons we shall need some additional conditions concerning the behavior of $P(z) = A(L(z))$, which are summarized in Section 4.

We should mention that the number of probability distributions meeting our constraints is quite limited due to the required independence. An example of a suitable model is based on an interarrival distribution with the so-called memoryless property, i.e., an exponential or geometric distribution, leading to (well-thumbed) Poisson- or Bernoulli-type arrivals within a cycle, respectively.

2. Tree Approach

In this section we will establish a one-to-one mapping between busy periods and a family of (labeled) planted planar trees, which provides a straightforward correspondence between deadline constraints and limited label sums of some subtrees. Due to this fact, we may relate the original problem of investigating the random variable $\text{SRD}(T)$ to a counting problem regarding a special (sub)family $\mathscr{B}_T$ of trees.

Consider the following diagram, which represents an example busy period:

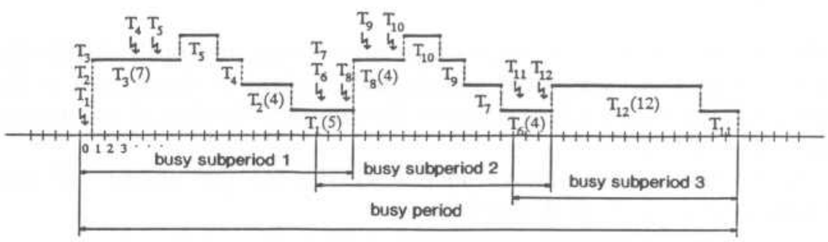


Fig. 1.

According to our discrete time model, the horizontal axis is divided into equidistant cycles. Those cycles forming the busy period of interest are numbered consecutively; cycle 0 denotes the initial (idle) cycle. Task arrivals are shown by small lightnings with task-names above. The execution of a task is displayed by a horizontal line whose length equals the task execution time. The vertical level of a line, i.e., its vertical distance to the horizontal axis, represents the number of tasks not processed to completion at the beginning of that task. For the sake of readability, each such line is marked with the name of the corresponding task (and, sometimes, its task execution time).

There is an important relation between deadline constraints and the length of the so-called *busy subperiods*. A busy subperiod is the epoch from the arrival of the first (new) task during the execution of a level 1 (or level 0) task to the end of the last cycle of the new task. For instance, looking at the cycle 0 in our example, one obtains the arrival of task T_1. Due to the nonpreemptive LCFS scheduling discipline, this task is badly off, because all tasks arriving before the beginning of the execution of T_1, are preferred! Hence, if the length of a busy subperiod is less or equal to T, all processed tasks are guaranteed to meet a service time deadline of T cycles. Conversely, if the length of a busy subperiod is larger than T, at least the task having arrived first will miss its deadline.

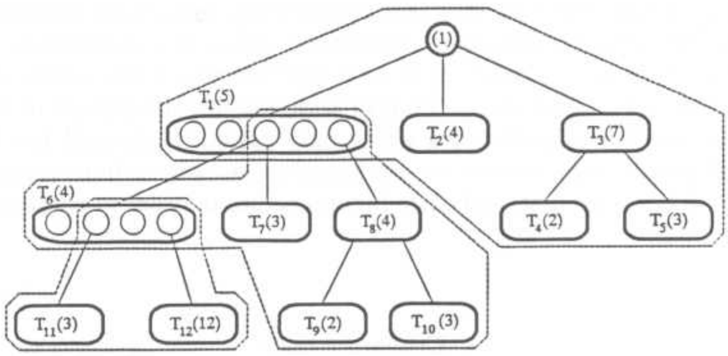


FIG. 2.

Consider the tree corresponding to the diagram above. To obtain this tree, a task is represented by an elliptical node which is labeled according to its task execution time; i.e., the label of a node is the number of cycles necessary for processing the task to completion. Equivalently, this labeling may be done by drawing the corresponding number of circles (each describing a cycle, of course) within the node.

The number of successors of a node equals the number of arrivals during the execution of the corresponding task. If a task has a task execution time of l cycles and will be scheduled to start at the ith cycle ($i \geq 0$) of the busy period, the execution will be completed at the end of the $(i + l - 1)$th cycle, since we are dealing with nonpreemptive LCFS scheduling. Thus, with t_i, $i \geq 0$, denoting the number of task arrivals during the ith cycle of the busy period, the number of successors of the node is $t_i + t_{i+1} + \cdots + t_{i+l-1}$. Successors are drawn from the left to the right, according to their arrival sequence.

Due to our construction, the outer leftmost (elliptical) nodes in the tree correspond to those tasks which both complete a busy subperiod and start a new one, too. They are displayed in the equivalent labeling style mentioned above. If such a node has no successors, it indicates the end of the whole busy period; at least one idle cycle follows.

Note that the reconstruction of the busy period from a given tree is done by a right-to-left preorder traversal of all (elliptical) nodes of the tree.

Deadline constraints are reflected by suitable limits on the number of cycles. More precisely, the sum of the labels of nodes belonging to a busy subperiod has to be less than the deadline T, for all busy subperiods, of course. In our example tree above, those nodes belonging to a specific busy subperiod are fenced in by a dotted line.

Unfortunately, the fact that consecutive busy subperiods overlap one another introduces unpleasant difficulties. Since two consecutive busy subperiods are pasted together at an outer leftmost node, (some of) its cycles have to be taken into account in both. On the other hand, to obtain the total number of cycles of a whole busy period, each cycle has to be counted exactly once. Hence, we are forced to investigate trees representing busy subperiods first, and paste them together in order to obtain whole busy periods.

3. Combinatorics

As mentioned in Section 1, a run denotes a sequence of busy periods not violating any task's deadline, followed by a busy period with at least one deadline violation. Let

$$b_{k,T} = \text{prob}\{\text{Length of a nonviolating busy period equals } k \text{ cycles}\}$$

and let

$$B_T(u) = \sum_{k \geq 0} b_{k,T} u^k \tag{3.1}$$

be the corresponding PGF. The PGF of the random variable SRD(T), i.e., the length of a successful run, is given by

$$S_T(u) = \sum_{k \geq 0} s_{k,T} u^k = \frac{1 - B_T(1)}{1 - B_T(u)}. \tag{3.2}$$

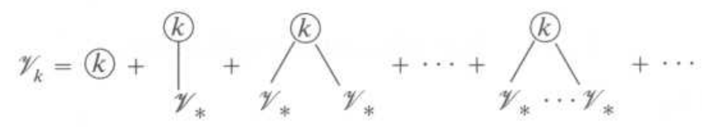
This follows from the fact that the PGF of the length of an arbitrary number of nonviolating busy periods is $\sum_{n \geq 0} B_T(u)^n$, and that the probability of the occurrence of the terminating violation busy periods equals $1 - B_T(1)$.

As promised, we start our treatment concerning $B_T(u)$ with studying the family $\mathcal{B}_{i,j}$ of trees which correspond to busy subperiods starting with a label i node and finishing with a label j node ($i \geq 1$, $j \geq 1$). We shall use symbolic equations for the description of classes of combinatorial structures (i.e., families of trees); cf. [3] for an overview. To keep the notation simple, we defer attaching the necessary weights to the translation into generating functions.

We have the following decomposition:

$$\mathcal{B}_{i,j} = \mathcal{H}_j\mathcal{V}_{i-1} + \mathcal{E}\mathcal{H}_j\mathcal{V}_{i-2} + \mathcal{E}^2\mathcal{H}_j\mathcal{V}_{i-3} + \cdots + \mathcal{E}^{i-2}\mathcal{H}_j\mathcal{V}_1 + \mathcal{E}^{i-1}\mathcal{H}_j. \tag{3.3}$$

The combinatorial objects used for building blocks have straightforward meanings. $\mathcal{E}$ denotes a single cycle with no task arrivals; $\mathcal{H}_j$ denotes a single cycle with at least one arrival, leading to the leftmost label j node. $\mathcal{V}_k$ denotes a sequence of $k \geq 1$ consecutive cycles with an arbitrary number of arrivals. To start with the most important one, we have the symbolic equation



with $\mathcal{V}_* = \sum_k \mathcal{V}_k$. In order to translate the symbolic equation into an appropriate ordinary generating function (OGF), we have to attach suitable sizes and weights to each combinatorial object. If we attach sizes by "multiplying" each elementary object (i.e., a node with label l) by z^l, the size of a (composed) object (i.e., a tree) is the sum of its labels. Additionally, providing suitable probability weights leads to an equivalent of the OGF of a class of combinatorial structures (i.e., a family of trees), namely the PGF of the random variable to which it corresponds.

For example, recalling Definition (1.2), the OGF of $\mathscr{V}_*$ reads

$$V_*(z) = \sum_{k\geq 1} l_k V_k(z).$$

Due to the definition of the PGF of task arrivals within a cycle, we have

$$q_{n,k} = \text{prob}\{n \text{ task arrivals during } k \text{ (consecutive) cycles}\} = [z^n]A(z)^k$$

for $n \geq 0$, $k \geq 1$. Obviously, $[z^n]f(z)$ denotes the coefficient of z^n in the power series expansion of $f(z)$. Thus, the OGF of $\mathscr{V}_k$ reads

$$\begin{aligned} V_k(z) &= q_{0,k} z^k + q_{1,k} z^k V_*(z) + \cdots + q_{n,k} z^k V_*(z)^n + \cdots \\ &= z^k \sum_{n\geq 0} q_{n,k} V_*(z)^n. \end{aligned}$$

Introducing the bivariate generating function

$$G(z,u) = \sum_{k\geq 0} l_k V_k(z) u^k,$$

one obtains $V_*(z) = G(z,1)$. Multiplying the above equation for $V_k(z)$ by $l_k u^k$ and summing up for $k \geq 1$ yields

$$\begin{aligned} G(z,u) &= \sum_{k\geq 0} l_k (zu)^k \sum_{n\geq 0} q_{n,k} G(z,1)^n \\ &= \sum_{n\geq 0} G(z,1)^n [w^n] \sum_{k\geq 1} l_k (A(w) zu)^k \\ &= L(zuA(G(z,1))). \end{aligned}$$

Because

$$V_1(z) = [l_1 u^1] G(z,u) = zA(G(z,1)),$$

we find that

$$\sum_{k\geq 1} l_k V_k(z) u^k = G(z,u) = L(uV_1(z)) = \sum_{k\geq 1} l_k V_1(z)^k u^k,$$

hence $V_k(z) = V_1(z)^k$ and $V_*(z) = G(z,1) = L(V_1(z))$. Substituting the latter in the equation for $V_1(z)$ above, we obtain

$$V_1(z) = zA(L(V_1(z))) = zP(V_1(z)).$$

As we might have expected, this is the generating function of a family of simply generated trees, cf. [6]. This function appeared in our investigations concerning preemptive LCFS scheduling ($B(z)$, see [8]) also.

At next, we look at $\mathscr{H}_j$, $j \geq 1$. The symbolic equation reads

with $\mathscr{T}_j$ denoting a label j node. Obviously, the corresponding OGF is $T_j(z) = l_j z^j$.

Since each combinatorial object in $\mathscr{H}_j$ corresponds to an object in $\mathscr{V}_1$, where the leftmost successor $\mathscr{V}_*$ (at the top level) is replaced by $\mathscr{T}_j$, we may omit the detailed translation of the symbolic equation and write down the result immediately:

$$H_j(z) = l_j z^j \frac{V_1(z) - a_0 z}{L(V_1(z))}.$$

Note that the term $a_0 z$ corresponds to the "smallest" tree in $\mathscr{V}_1$, which consists of the root only (no arrivals during the corresponding cycle).

The OGF for $\mathscr{E}$ is straightforward; mentioning Definition (1.2), we have

$$E(z) = a_0 z.$$

We summarize the results in the following lemma.

Lemma 3.1. *With the notations above, the ordinary generating functions of* $\mathscr{V}_k$, $\mathscr{H}_j$, *and* $\mathscr{E}$, *respectively, are given by*

$$V_k(z) = B(z)^k \qquad \text{for } k \geq 1,$$

$$H_j(z) = l_j z^j \frac{B(z) - a_0 z}{L(B(z))} \qquad \text{for } j \geq 1,$$

$$E(z) = a_0 z,$$

where $B(z)$ *denotes the solution of*

$$B(z) = zA(L(B(z))) = zP(B(z)).$$

Now we are able to translate the symbolic equation (3.3) into the appropriate PGF. For reasons which will become more clear when pasting busy subperiods together, we shall attach two different sizes to a structure from that class. Roughly speaking, the size represented by z is responsible for counting the length of the corresponding busy subperiod w.r.t. deadline properties. A different size is represented by the variable u. It counts the contributions of the corresponding busy subperiod to the overall length of the whole busy period; remember our remarks at the end of Section 2. We find that

$$B_{i,j}(z,u) = \sum_{l=0}^{i-1} E(1)^{i-1-l} H_j(zu) B(zu)^l u^{-1-l}.$$

Note that we should have no contributions from $\mathscr{E}$, for either deadline counting or the overall size; thus $E(1)$ is used. The last term u^{-1-l} makes the difference in the size counted by z and u. The $l+1$ cycles within the initial label i node, i.e., the "roots" of $\mathscr{H}_j$ and $\mathscr{V}_1^l$, must be counted in z only (deadlines), not in u. The latter is done in the preceding busy subperiod!

That is, for a busy subperiod starting with a label i and terminating with a label j node, $[z^t][u^n]B_{i,j}(z,u)$ is the probability that all tasks meet a deadline of t cycles (and no smaller one), contributing n cycles to the length of the whole busy period.

Simplifying the expression above yields

$$\begin{aligned} B_{i,j}(z,u) &= \frac{B(zu) - a_0 zu}{L(B(zu))} l_j(zu)^j \sum_{l=0}^{i-1} a_0^{i-1-l} B(zu)^l u^{-1-l} \\ &= \left(\left(\frac{B(zu)}{u} \right)^i - a_0^i \right) \frac{B(zu) - a_0 zu}{L(B(zu))} \cdot \frac{1}{B(zu) - ua_0} l_j(zu)^j. \end{aligned} \tag{3.4}$$

Now, we shall try to paste busy subperiods together. In order to allow deadline counting in each busy subperiod, we are forced to use *different counting variables* z_k instead of z. Let $\mathscr{B}_{i,j}^k$ denote the family of trees, which are formed by pasting together exactly $k \geq 1$ busy subperiods. For example, we have

$$\mathscr{B}_{i,j}^2 = \sum_{k \geq 1} \mathscr{B}_{i,k} \mathscr{B}_{k,j};$$

the corresponding (multivariate) generating function reads

$$B_{i,j}^2(z_2, z_1; u) = \sum_{k \geq 1} B_{i,k}(z_2, u) B_{k,j}(z_1, u).$$

To keep notations simple, we introduce the abbreviations

$$B_{i,j}(z,u) = S_i(z,u) I(z,u) T_j(z,u),$$

cf. Equation (3.4). Using these, we obtain

$$B^2_{i,j}(z_2, z_1; u) = S_i(z_2, u)I(z_2, u) \sum_{k\geq 1} T_k(z_2, u)S_k(z_1, u)I(z_1, u)T_j(z_1, u)$$

$$= S_i(z_2, u)I(z_2, u) \sum_{k\geq 1} l_k(z_2 u)^k \left(\left(\frac{B(z_1 u)}{u}\right)^k - a_0^k\right)$$

$$\cdot I(z_1, u)T_j(z_1, u)$$

$$= S_i(z_2, u)I(z_2, u)\left[L(z_2 B(z_1 u)) - L(a_0 z_2 u)\right] I(z_1, u)T_j(z_1, u).$$

Note that overlapping of busy subperiods is reflected by the "connecting function" within the brackets. The starting and trailing functions $S_i(\cdot, u)$ and $T_j(\cdot, u)$ appear in the expression again; thus we may use this technique repeatedly to construct the general term:

$$B^k_{i,j}(z_k, \ldots, z_1; u) = S_i(z_k, u)I(z_k, u)$$

$$\cdot (L(z_k B(z_{k-1}u)) - L(a_0 z_k u))I(z_{k-1}, u)$$

$$\vdots$$

$$\cdot (L(z_2 B(z_1 u)) - L(a_0 z_2 u))I(z_1, u)$$

$$\cdot T_j(z_1, u).$$

To construct a whole busy period consisting of exactly k busy subperiods, we have to deal with the decomposition

$$\mathscr{C}^k = \mathscr{U} \sum_{j\geq 1} \mathscr{B}^k_{1,j}\mathscr{E}_j.$$

$\mathscr{U}$ denotes a single cycle forming the initial cycle of the first busy subperiod; its OGF is $U(z) = z$. $\mathscr{E}_j$ is a label j node with no arrivals; according to Lemma 3.1, we obtain the OGF $E_j(z) = E(z)^j = (a_0 z)^j$. Translating the symbolic equation above, we find

$$\mathscr{C}^k(z_k, \ldots, z_1; u) = u \sum_{j\geq 1} B^k_{1,j}(z_k, \ldots, z_1; u)E(1)^j.$$

Note that we do not count cycles resulting from the terminating idle period, i.e., $\mathscr{E}_j$. We easily obtain

$$
\begin{aligned}
C^k(z_k,\dots,z_1;u) &= uS_1(z_k,u)I(z_k,u)\\
&\quad\cdot\big(L(z_kB(z_{k-1}u)) - L(a_0z_ku)\big)I(z_{k-1},u)\\
&\quad\vdots\\
&\quad\cdot\big(L(z_2B(z_1u)) - L(a_0z_2u)\big)I(z_1,u)\\
&\quad\cdot L(a_0z_1u)\\
&= u\left(\frac{B(z_ku)}{u} - a_0\right)\frac{B(z_ku) - a_0z_ku}{L(B(z_ku))}\cdot\frac{1}{B(z_ku) - ua_0}\\
&\quad\cdot\big(L(z_kB(z_{k-1}u))\\
&\qquad\quad - L(a_0z_ku)\big)\frac{B(z_{k-1}u) - a_0z_{k-1}u}{L(B(z_{k-1}u))}\\
&\quad\cdot\frac{1}{B(z_{k-1}u) - ua_0}\\
&\qquad\vdots\\
&\quad\cdot\big(L(z_2B(z_1u)) - L(a_0z_2u)\big)\frac{B(z_1u) - a_0z_1u}{L(B(z_1u))}\\
&\quad\cdot\frac{1}{B(z_1u) - ua_0}\\
&\quad\cdot L(a_0z_1u).
\end{aligned}
$$

Obviously, a busy period with no busy subperiods, that is, an idle cycle, has the symbolic equation $\mathscr{U}\mathscr{E}_1$. The corresponding OGF is very simple:

$$C^0(u) = a_0u.$$

Since a whole busy period may consist of an arbitrary number of busy subperiods not exceeding T cycles (for deadline counting, of course), we are forced to study

$$
\begin{aligned}
B_T(u) = a_0u + \sum_{k\ge 1}\left[z_k^T\right]\cdots\left[z_1^T\right]\frac{1}{1-z_k}\\
\cdot\frac{1}{1-z_{k-1}}\cdots\frac{1}{1-z_1}C^k(z_k,\dots,z_1;u),
\end{aligned}
$$

which is the PGF of the length of an arbitrary busy period containing no deadline violation. Introducing the abbreviations $y_k = z_k u$ and

$$\begin{aligned}
D^k(y_k, \dots, y_1; u) &= \frac{1}{1 - y_k/u} \cdot \frac{1}{1 - y_{k-1}/u} \cdots \frac{1}{1 - y_1/u} C^k(y_k/u, \dots, y_1/u; u) \\
&= \frac{1}{1 - y_k/u} \cdot \frac{B(y_k) - a_0 y_k}{L(B(y_k))} \\
&\quad \cdot \frac{1}{1 - y_{k-1}/u} \cdot \frac{L((y_k/u)B(y_{k-1})) - L(a_0 y_k)}{B(y_{k-1}) - u a_0} \\
&\quad \cdot \frac{B(y_{k-1}) - a_0 y_{k-1}}{L(B(y_{k-1}))} \\
&\quad \vdots \\
&\quad \cdot \frac{1}{1 - y_1/u} \cdot \frac{L((y_2/u)B(y_1)) - L(a_0 y_2)}{B(y_1) - u a_0} \\
&\quad \cdot \frac{B(y_1) - a_0 y_1}{L(B(y_1))} \\
&\quad \cdot L(a_0 y_1)
\end{aligned}$$

we find

$$B_T(u) = a_0 u + \sum_{k \geq 1} u^{kT} [y_k^T] \cdots [y_1^T] D^k(y_k, \dots, y_1; u). \tag{3.6}$$

4. Asymptotics

Looking more closely at the (delicate) expression for $B_T(u)$, one obtains nontrivial interdependencies among the coefficients $[y_k^T], \dots, [y_1^T]$, resulting from the "connecting functions" $L((y_i/u)B(y_{i-1}))$. Hence, a direct extraction of the desired coefficients yields terribly complicated expressions, at first (and possibly second) sight far away from tractability. Thus, we shall use the powerful tool of singularity analysis of generating functions instead; see [3] for a survey.

Such techniques are based on the fact that the asymptotic behavior of a Taylor coefficient $[z^n]f(z)$ of an analytic function largely depends on the behavior of $f(z)$ near its dominant singularities, i.e., the ones of smallest modulus. In fact, restricting ourselves to functions having only one singularity $z = \zeta$ on their circle of convergence, it is possible to deal with local expansions of $f(z)$ near ζ. For instance, if $f(z) = O(g(z))$ for $z \to \zeta$, we have $[z^n]f(z) = O([z^n]g(z))$ for $n \to \infty$, under fairly general conditions;

consider [4] for a rigorous treatment concerning suitable "scales" of functions $g(z)$ and appropriate *transfer lemmas*. An example is the following

LEMMA 4.1 (Transfer Lemma). *Assume that, with the sole exception of the (real and positive) singularity $z = \zeta > 0$, the function $f(z)$ is analytic in the indented disk $\Delta_\zeta(\eta, \varphi) = \{z\colon |z| \le \zeta + \eta,\ |\arg(z - \zeta)| \ge \varphi,\ z \ne \zeta\}$, where $\eta > 0$ and $0 < \varphi < \pi/2$. Assume further that as z tends to ζ in $\Delta_\zeta(\eta, \varphi)$,*

$$f(z) = O\big((1 - z/\zeta)^\alpha\big)$$

for some real number α. Then the nth Taylor coefficient of $f(z)$ satisfies

$$[z^n]f(z) = O(n^{-\alpha-1}\zeta^{-n}).$$

Note that this lemma requires analytic continuation of $f(z)$ beyond its circle of convergence, but only order of growth information (and no side conditions à la Tauber or Darboux). The proof is based on estimations of Cauchy's formula using a suitable contour in $\Delta_\zeta(\eta, \varphi)$; see [4] for details.

Using Lemma 4.1, we shall establish the (well-known, cf. [6]) asymptotic behavior of the nth Taylor coefficient of

$$B(z) = \sum_{n \ge 0} b_n z^n,$$

which denotes the (nonnegative, that is, $b_n \ge 0$) solution of the functional equation $B(z) = zP(B(z))$ with $P(w) = A(L(w))$, cf. Lemma 3.1. The following conditions are to be met:

(i) $A(0) > 0$, cf. Section 2.

(ii) $L(0) = 0$, cf. Section 2.

(iii) $P''(1) > 0$, which implies that for some $l \ge 2$

$$P(w) = \sum_{n \ge 0} p_n w^n = p_0 + p_1 w + p_l w^l + O(w^{l+1}) \qquad \text{for } w \to 0,$$

where $p_0 = a_0 > 0$ and $p_l > 0$.

(iv) The equation $P(w) - wP'(w) = 0$ has a real solution $w = \tau > 1$. Note that this forces $P'(1) < 1$, as can be shown by simple geometric arguments.

(v) $P(w)$ has a radius of convergence larger than τ.

(vi) $L(w)$ has a radius of convergence larger than $\tau^2/P(\tau)$. This condition will become meaningful later in this section.

Providing this, we are able to state the following

LEMMA 4.2 (Expansion of $B(z)$). *With the notations and conditions above, the function* $B(z)$ *is analytic in a domain* $\Delta_\rho(\eta, \varphi)$ *with some* $\eta > 0$ *and* $0 < \varphi < \pi/2$. *There is only one algebraic singularity* $z = \rho$ *on its circle of convergence, in whose neighborhood*

$$B(z) = \tau - b \cdot (1 - z/\rho)^{1/2} + O(1 - z/\rho) \qquad \text{for } z \to \rho,\ z \in \Delta_\rho(\eta, \varphi)$$

with $b = \sqrt{2P(\tau)/P''(\tau)}$. *The asymptotic expansion of* $b_n = [z^n]B(z)$ *reads*

$$b_n = \frac{b}{2\sqrt{\pi}} n^{-3/2} \rho^{-n} + O(n^{-2}\rho^{-n}) \qquad \text{for } n \to \infty.$$

Proof. Let

$$F(z,w) = \sum_{i+j \geq 1} f_{ij} z^i w^j = zP(w).$$

The following facts are easily established:

(1) All f_{ij} (and b_n, of course) are nonnegative and condition (iii) implies $f_{ij} > 0$ for some $j \geq 2$. In addition, $f_{00} > 0$ by condition (i) and $f_{01} = 0 \neq 1$.

(2) Let

$$\rho = \frac{\tau}{P(\tau)}.$$

We have $1 < \rho < \tau$. The lower bound follows from the fact that $f(x) = x/P(x)$ is strictly monotonic for $x < \tau$ and $f(1) = 1$; the upper bound is obvious since $P(\tau) > 1$. It is clear from condition (v) that the point (ρ, τ) lies within the region where $F(z,w)$ converges (absolutely) and that

$$F(\rho, \tau) = \tau$$
$$F_w(\rho, \tau) = 1;$$

remember condition (iv).

(3) There is some $j > i \geq 1$ with $\gcd(i,j) = 1$ such that $b_i b_j > 0$. Provided that $B(z)$ exists, we have $b_0 = B(0) = 0 \cdot P(B(0)) = 0$, hence $B(z) = O(z)$ for $z \to 0$. Bootstrapping yields

$$B(z) = zP(O(z)) = zp_0 + O(z^{m+1}) \qquad \text{for } z \to 0$$

with $m \geq 1$ by virtue of condition (iii). A second step establishes

$$B(z) = zP(zp_0 + O(z^{m+1})) = zp_0 + p_m p_0^m z^{m+1} + O(z^{m+2}) \quad \text{for } z \to 0.$$

Hence, $i = 1$ and $j = m + 1 \geq 2$ provide the required result.

At this point, all conditions necessary for the application of Theorem 2 of [7] are established. This provides the conclusion that $B(z)$ is analytic and has an algebraic singularity $z = \rho$ on its circle of convergence (but no others) and $B(\rho) = \tau$. The required expansion for $z \to \rho$ follows from Theorem 5 of [1]. Since $B(z)$ may be continued analytically beyond its circle of convergence for all $z = \zeta$ with $|\zeta| = \rho$ but $\zeta \neq \rho$, as can be shown using the Implicit Function Theorem, $B(z)$ is analytic in a domain $\Delta_\rho(\eta, \varphi)$ with $\eta > 0$ and $0 < \varphi < \pi/2$. The same is true for $g(z) = -(1 - z/\rho)^{1/2}$, whose nth Taylor coefficient g_n has the well-known expansion

$$g_n = \frac{1}{2\sqrt{\pi}} n^{-3/2}\rho^{-n} + O(n^{-5/2}\rho^{-n}) \qquad \text{for } n \to \infty.$$

Thus, we are permitted to apply Lemma 4.1 to $B(z) - \tau + b \cdot (1 - z/\rho)^{1/2}$, which establishes the asymptotic expansion of b_n as asserted. ∎

Keep in mind that the remainder $O(1 - z/\rho)$ in the expansion of $B(z)$ represents a function which is analytic in $\Delta_\rho(\eta, \varphi)$! By the way, it is easy to improve this remainder term to $O((1 - z/\rho)^{3/2})$ by using the more accurate expansion $B(z) = \tau - b \cdot (1 - z/\rho)^{1/2} + c \cdot (1 - z/\rho) + O((1 - z/\rho)^{3/2})$ for $z \to \rho$, $z \in \Delta_\rho(\eta, \varphi)$. The linear term contributes nothing to the nth Taylor coefficient, hence

$$b_n = \frac{b}{2\sqrt{\pi}} n^{-3/2}\rho^{-n} + O(n^{-5/2}\rho^{-n}) \qquad \text{for } n \to \infty. \tag{4.1}$$

After this preliminary discussion we shall give a short and informal overview of how to proceed with the investigation of $B_T(u)$. Since we are interested in (factorial) moments of SRD(T), that is, derivatives of $S_T(u)$ at $u = 1$ (cf. Eq. (3.2)), we have to deal with derivatives of $B_T(u)$ evaluated at $u = 1$. Thus, in what follows, we assume u to be a complex parameter in the closed disk $D(1, \nu) = \{z\colon |z - 1| \leq \nu\}$ for some arbitrary small $\nu > 0$.

Looking more closely at y_1-related terms in $B_T(u)$, that is,

$$\frac{1}{1 - y_1/u} \cdot \frac{L((y_2/u)B(y_1)) - L(a_0 y_2)}{B(y_1) - ua_0} \cdot \frac{B(y_1) - a_0 y_1}{L(B(y_1))} \cdot L(a_0 y_1), \tag{4.2}$$

our task is the determination of the Tth Taylor coefficient $[y_1^T]$ in this multivariate function, which is analytic for y_1, y_2 in a neighborhood of 0 and $u \in D(1, \nu)$. Due to general theorems (Cauchy's formula for multivariate analytic functions), $[y_1^T]f(y_2, y_1, u)$ is an analytic function of y_2 and u, also. In addition, it is not hard to prove that the statement of Lemma 4.1 remains valid for a multivariate analytic function. For example, if

$$f(z,w) = O\big(g(w)(1 - z/\zeta)^{\alpha}\big) \qquad \text{for } z \to \rho$$

uniformly w.r.t. w, it follows that

$$[z^n]f(z,w) = O\big(g(w)n^{-1-\alpha}\zeta^{-n}\big) \qquad \text{for } n \to \infty$$

uniformly in w, also. Again, keep in mind that the latter $O(\cdot)$ represents a function which is analytic in w!

Returning to our original function, we obtain three "sources" of singularities,

(1) a (removeable) simple pole at $y_1 = \zeta(u) < 1$, resulting from $B(\zeta(u)) = a_0 u$,

(2) a simple pole at $y_1 = u$,

(3) an algebraic singularity at $y_1 = \rho$ resulting from functions involving $B(y_1)$.

The fact that $y_1 = \zeta(u)$ is a removeable singularity, i.e., that there is no singularity at all, is easily established by taking into account the zero of $L((y_2/u)B(y_1)) - L(a_0 y_2)$ at $y_1 = \zeta(u)$.

Remembering $\rho > 1$ it follows that $y_1 = u$ is the singularity with the smallest modulus; in fact we choose ν small enough, i.e., $1 + \nu < \rho$. The appropriate contribution to $[y_1^T]$ is easily determined via *subtracted singularities*:

$$\frac{L((y_2/u)B(u)) - L(a_0 y_2)}{L(B(u))} L(a_0 u) \cdot u^{-T}.$$

Investigating the behavior of (4.2) near the "next" singularity $y_1 = \rho$ it turns out that $B(y_1) - a_0 y_1$ and $L(B(y_1))$ obey expansions similar to

$B(y_1)$. The fuction $L(a_0 y_1)$ has a radius of convergence larger than ρ by virtue of condition (vi), i.e., is well behaved in a neighborhood of $y_1 = \rho$. Hence, the only remaining difficulty concerns the term containing the "connecting function," i.e.,

$$\frac{L((y_2/u)B(y_1)) - L(a_0 y_2)}{B(y_1) - ua_0}.$$

But, using the mentioned extension of our devices, it is possible to attack this multivariate analytic function, too. Since y_1 comes up with $B(y_1)$, one feels that $L((y_2/u)B(y_1)$ should have an algebraic singularity at $y_1 = \rho$, *independent* of y_2! Due to the fact that, at our next "stage," y_2 will play the role of y_1 and y_3 that of y_2, it is obvious to ask for the behavior in a neighborhood of $y_2 = \rho$ (and $y_2 = u$, resulting from the subtracted singularity term for y_2, too). However, since y_2 appears in conjunction with the well-behaved function $L(\cdot)$ only, we may expect inferior influences here. To make a long story short, we assert that it is possible to determine a uniform expansion

$$\frac{L((y_2/u)B(y_1)) - L(a_0 y_2)}{B(y_1) - ua_0} = b(y_2, u) + c(y_2, u)(1 - y_1/\rho)^{1/2}$$
$$+ O(1 - y_1/\rho) \qquad \text{for } y_1 \to \rho,$$

where $b(y_2, u)$ and $c(y_2, u)$ denote well-behaved analytic functions of both y_2 and u. The remainder $O(1 - y_1/\rho)$ represents a multivariate analytic function, too, and the implied constant is independent of y_1, y_2 and u.

Note that although it is impossible to separate the "connecting function" directly, i.e., to split up $L((y_2/u)B(y_1))$ into a product $f(y_1)g(y_2)$, an *asymptotic separation* succeeded!

Putting all terms together, we obtain a uniform expansion for (4.2) at $y_1 = \rho$, similar to the expansion above:

$$\beta(y_2, u)L(a_0\rho) + \gamma(y_2, u)L(a_0\rho)(1 - y_1/\rho)^{1/2} + O(1 - y_1/\rho)$$
$$\text{for } y_1 \to \rho.$$

Note that the terms $L(a_0\rho)$ represent the contribution resulting from the terminating function $L(a_0 y_1)$.

The subtracted term resulting from the simple pole $y_1 = u$ is meaningless for the analysis of the singularity $y_1 = \rho$ since $(1 - y_1/u)^{-1}$ is analytic for all $y_1 \neq u$. Using transfer lemmas, the desired coefficient $[y_1^T]$ finally yields

$$\alpha(y_2, u)L(a_0 u)u^{-T} - \frac{1}{2\sqrt{\pi}}\gamma(y_2, u)L(a_0\rho)T^{-3/2}\rho^{-T} + O(T^{-2}\rho^{-T})$$

for $T \to \infty$

with both $\alpha(y_2, u)$ and $\gamma(y_2, u)$ analytic at $y_2 = \rho$; the "elimination" of y_1 is complete.

Now, the same procedure may be used for the extraction of $[y_2^T]$ (hence for all $[y_i^T]$), since the related terms are almost the same. In fact, the only difference springs from replacing $L(a_0 y_1)$ by $\alpha(y_2, u)$ and $\gamma(y_2, u)$, respectively! Using this simple iterative scheme (leading to a recurrence relation) it is possible to compute an asymptotic expansion

$$B_T(u) = B(u) - R(u)u^T T^{-3/2}\rho^{-T} + O(u^T T^{-2}\rho^{-T})$$

uniformly valid for $u \in D(1, \nu)$. By virtue of a general theorem concerning uniform expansions we may differentiate this expansion in order to derive $B_T^{(m)}(1)$ for an arbitrary but fixed m.

We start our detailed treatment with providing some "building blocks," that is, asymptotic expansions of the functions involved. At first we look at $C(z) = L(B(z))$, which obviously denotes the (positive) solution of $C(z) = L(zA(C(z)))$. Thus, the same procedure as in the proof of Lemma 4.2 might be used. This would establish that $C(z)$ has exactly one algebraic singularity $z = \rho$ on its circle of convergence and $C(\rho) = L(\tau)$, i.e., provides an asymptotic expansion similar to $B(z)$.

But, since we need the uniform asymptotic expansion of $C(z, w) = L(wB(z))$ for an arbitrary *complex* value $w \in \Delta_\rho(\eta, \varphi)$, too (which covers $C(z)$ above), we shall use another idea. Note, however, that $C(z, w)$ is a solution of $C(z, w) = L(wzP(L^{[-1]}(C(z, w))/w))$ which (formally) leads to an algebraic singularity $z = \rho$ and $C(\rho, w) = L(w\tau)$.

Our alternative approach is based on condition (vi), which guarantees that $v = w\tau$ lies within the radius of convergence of $L(\cdot)$ for all $w \in D(0, \rho + \varepsilon)$ for some $\varepsilon > 0$ sufficiently small. Using the Taylor expansion at $w\tau$, i.e.,

$$L(v) = \sum_{n \geq 0} \frac{L^{(n)}(w\tau)}{n!}(v - w\tau)^n,$$

valid for $v \in D(w\tau, \varepsilon)$, and substituting

$$v = wB(z) = w\tau - wb(1 - z/\rho)^{1/2}(1 + r(z))$$

with $r(z) = O(1 - z/\rho)^{1/2}$ (analytic for $z \in \Delta_\rho(\eta, \varphi)$), we have for z sufficiently close to ρ

$$\begin{aligned} L(wB(z)) &= \sum_{n \ge 0} \frac{L^{(n)}(w\tau)}{n!}\left(-wb(1 - z/\rho)^{1/2}(1 + r(z))\right)^n \\ &= L(w\tau) - bwL'(w\tau)(1 - z/\rho)^{1/2} + O(1 - z/\rho) \\ &\qquad \text{for } z \to \rho,\ z \in \Delta_\rho(\eta, \varphi), \end{aligned}$$

where the $O(\cdot)$-term is uniformly valid for $w \in D(0, \rho + \varepsilon)$ (and denotes an analytic function of both z and w, of course)! Using this result, we find that for $y_{k-1} \to \rho$[1]

$$\begin{aligned} \frac{1}{L(B(y_{k-1}))} &= \frac{1}{L(\tau) - bL'(\tau)(1 - y_{k-1}/\rho)^{1/2}\left(1 + O\left((1 - y_{k-1}/\rho)^{1/2}\right)\right)} \\ &= \frac{1}{L(\tau)} + \frac{bL'(\tau)}{L(\tau)^2}(1 - y_{k-1}/\rho)^{1/2} + O(1 - y_{k-1}/\rho) \end{aligned}$$

and

$$\begin{aligned} L\left(\frac{y_k}{u}B(y_{k-1})\right) &= L\left(\frac{y_k}{u}\tau\right) - b\frac{y_k}{u}L'\left(\frac{y_k}{u}\tau\right)(1 - y_{k-1}/\rho)^{1/2} \\ &\quad + O(1 - y_{k-1}/\rho) \end{aligned}$$

uniformly for $y_k \in D(0, \rho + \varepsilon)$ and $u \in D(1, \nu)$. The latter and Lemma 4.2 yield the asserted expansion for

$$Q(y_k, y_{k-1}, u) = \frac{L((y_k/u)B(y_{k-1})) - L(a_0 y_k)}{B(y_{k-1}) - ua_0},$$

[1]In what follows we use the notation $y_{k-1} \to \rho$ as an abbreviation for $y_{k-1} \to \rho$, where $y_{k-1} \in \Delta_\rho(\eta, \varphi)$.

namely

$$Q(y_k, y_{k-1}, u)$$

$$= \frac{L((y_k/u)\tau) - L(a_0 y_k) - b(y_k/u)L'((y_k/u)\tau)(1 - y_{k-1}/\rho)^{1/2} + O(1 - y_{k-1}/\rho)}{\tau - a_0 u - b(1 - y_{k-1}/\rho)^{1/2} + O(1 - y_{k-1}/\rho)}$$

$$= \frac{L((y_k/u)\tau) - L(a_0 y_k)}{\tau - a_0 u} + \left(\frac{b(L((y_k/u)\tau) - L(a_0 y_k))}{(\tau - a_0 u)^2} - \frac{b(y_k/u)L'((y_k/u)\tau)}{\tau - a_0 u} \right)(1 - y_{k-1}/\rho)^{1/2} + O(1 - y_{k-1}/\rho) \qquad \text{for } y_{k-1} \to \rho,$$

uniformly valid for $y_k \in D(0, \rho + \varepsilon)$ and $u \in D(1, \nu)$. Remember our remarks in the preceding overview, especially concerning the functions $b(y_2, u)$ and $c(y_2, u)$!

For the sake of completeness we mention the trivial expansions

$$\frac{1}{1 - y_{k-1}/u} = \frac{1}{1 - \rho/u} + O(1 - y_{k-1}/\rho),$$

$$B(y_{k-1}) - a_0 y_{k-1} = \tau - a_0 \rho - b(1 - y_{k-1}/\rho)^{1/2} + O(1 - y_{k-1}/\rho)$$

as $y_{k-1} \to \rho$, and

$$L(a_0 y_k) = L(a_0 \rho) + O(1 - y_k/\rho),$$

$$L\left(\frac{y_k}{u}\tau\right) = L\left(\frac{\rho}{u}\tau\right) + O(1 - y_k/\rho)$$

as y_k tends to ρ, uniformly for $u \in D(1, \nu)$.

Thus, all building blocks for the expansion of the expressions a la (4.2) are present. Let $t_{k-1}(y, u, T)$ denote a function analytic for $y \in D(0, \rho + \varepsilon)$ and $u \in D(1, \nu)$ with

$$t_{k-1}(y, u, T) = t_{k-1}(\rho, u, T)(1 + O(1 - y/\rho)) \qquad \text{for } y \to \rho \quad (4.3)$$

uniformly for $u \in D(1, \nu)$, $k \geq 1$, and $T \to \infty$. Starting with $t_1(y, u, T) = L(a_0 y)$ we shall investigate $F_k(y_k, y_{k-1}) = F_k(y_k, y_{k-1}, u, T)$ for $k \geq 2$, which reads

$$
\begin{aligned}
F_k(y_k, y_{k-1}) &= \frac{1}{1 - y_{k-1}/u} \cdot \frac{L((y_k/u)B(y_{k-1})) - L(a_0 y_k)}{B(y_{k-1}) - ua_0} \\
&\quad \cdot \frac{B(y_{k-1}) - a_0 y_{k-1}}{L(B(y_{k-1}))} t_{k-1}(y_{k-1}, u, T) \\
&= \beta(y_k, u) t_{k-1}(\rho, u, T) + \gamma(y_k, u) t_{k-1}(\rho, u, T) \\
&\quad \cdot (1 - y_{k-1}/\rho)^{1/2} \\
&\quad + O(t_{k-1}(\rho, u, T)(1 - y_{k-1}/\rho)) \qquad \text{as } y_{k-1} \to \rho,
\end{aligned}
$$

uniformly for $y_k \in D(0, \rho + \varepsilon)$, $u \in D(1, \nu)$, $k \geq 1$, and $T \to \infty$. Moreover, the functions $\beta(y_k, u)$ and $\gamma(y_k, u)$ are analytic and

$$
\beta(y_k, u) = \frac{1}{1 - \rho/u} \cdot \frac{L((y_k/u)\tau) - L(a_0 y_k)}{\tau - ua_0} \cdot \frac{\tau - a_0 \rho}{L(\tau)}
$$

and

$$
\begin{aligned}
\gamma(y_k, u) &= \frac{1}{1 - \rho/u} \\
&\quad \cdot \Bigg[b \left(\frac{L((y_k/u)\tau) - L(a_0 y_k)}{(\tau - ua_0)^2} - \frac{(y_k/u) L'((y_k/u)\tau)}{\tau - a_0 u} \right) \\
&\quad \cdot \frac{\tau - a_0 \rho}{L(\tau)} \\
&\quad - \frac{L((y_k/u)\tau) - L(a_0 y_k)}{\tau - ua_0} \cdot \frac{b}{L(\tau)} \\
&\quad + \frac{L((y_k/u)\tau) - L(a_0 y_k)}{\tau - ua_0} (\tau - a_0 \rho) \frac{bL'(\tau)}{L(\tau)^2} \Bigg] \\
&= \frac{b}{1 - \rho/u} \cdot \frac{\tau - a_0 \rho}{\tau - a_0 u} \\
&\quad \cdot \frac{1}{L(\tau)} \Bigg[\frac{L((y_k/u)\tau) - L(a_0 y_k)}{\tau - ua_0} \\
&\quad - \frac{y_k}{u} L'\left(\frac{y_k}{u} \tau \right) - \frac{L((y_k/u)\tau) - L(a_0 y_k)}{\tau - a_0 \rho} \\
&\quad + \frac{L'(\tau)}{L(\tau)} \left(L\left(\frac{y_k}{u} \tau \right) - L(a_0 y_k) \right) \Bigg]. \qquad (4.4)
\end{aligned}
$$

The contributions resulting from the simple pole at $y_{k-1} = u$ are easily determined via subtracted singularities, as already mentioned. The subtracted function is

$$S_k(y_k, y_{k-1}, u, T) = \frac{1}{1 - y_{k-1}/u}\alpha(y_k, u)t_{k-1}(u, u, T),$$

where

$$\alpha(y_k, u) = \frac{L((y_k/u)B(u)) - L(a_0 y_k)}{L(B(u))}. \tag{4.5}$$

Its Tth Taylor coefficient is an analytic function for $y_k \in D(0, \rho + \varepsilon)$ and $u \in D(1, \nu)$ and reads

$$\left[y_{k-1}^T\right]S_k(y_k, y_{k-1}, u, T) = \alpha(y_k, u)t_{k-1}(u, u, T)u^{-T}. \tag{4.6}$$

Obviously, $S_k(y_k, y_{k-1}, u, T)$ is analytic at $y_{k-1} = \rho$; hence

$$S_k(y_k, y_{k-1}, u, T) = \frac{1}{1 - \rho/u}\alpha(y_k, u)t_{k-1}(u, u, T)$$
$$+ O(t_{k-1}(u, u, T)(1 - y_{k-1}/\rho))$$

for $y_{k-1} \to \rho$, uniformly for $y_k \in D(0, \rho + \varepsilon)$, $u \in D(1, \nu)$, $k \geq 1$ and $T \to \infty$. Thus, the function $G_k(y_k, y_{k-1}, u, T) = F_k(y_k, y_{k-1}, u, T) - S_k(y_k, y_{k-1}, u, T)$ has no singularity at $y_{k-1} = u$ but the same singularity $y_{k-1} = \rho$ as $F_k(y_k, y_{k-1}, u, T)$. For $y_{k-1} \to \rho$ we obtain the expansion

$$G_k(y_k, y_{k-1}, u, T) = \beta(y_k, u)t_{k-1}(\rho, u, T)$$
$$- \frac{1}{1 - \rho/u}\alpha(y_k, u)t_{k-1}(u, u, T)$$
$$+ \gamma(y_k, u)t_{k-1}(\rho, u, T)(1 - y_{k-1}/\rho)^{1/2}$$
$$+ O(t_{k-1}(u, u, T)(1 - y_{k-1}/\rho))$$
$$+ O(t_{k-1}(\rho, u, T)(1 - y_{k-1}/\rho)),$$

uniformly valid for $y_k \in D(0, \rho + \varepsilon)$, $u \in D(1, \nu)$, $k \geq 1$ and $T \to \infty$. This finally establishes the asymptotic expansion of $[y_{k-1}^T]F_k(y_k, y_{k-1}, u, T)$;

using our Transfer Lemma, we obtain (that is, define) for $k \geq 2$

$$\begin{aligned} t_k(y_k, u, T) &= [y_{k-1}^T] F_k(y_k, y_{k-1}, u, T) \\ &= \alpha(y_k, u) t_{k-1}(u, u, T) u^{-T} \\ &\quad - \frac{1}{2\sqrt{\pi}} \gamma(y_k, u) t_{k-1}(\rho, u, T) T^{-3/2} \rho^{-T} \\ &\quad + O\big(t_{k-1}(u, u, T) T^{-2} \rho^{-T}\big) + O\big(t_{k-1}(\rho, u, T) T^{-2} \rho^{-T}\big) \end{aligned} \tag{4.7}$$

as T tends to infinity. Obviously, the first term springs up from the subtracted singularity term, cf. (4.6). Keep in mind that the (uniform) remainder terms represent bivariate analytic functions, say $R_k(y_k, u, T)$ and $\overline{R}_k(y_k, u, T)$, both analytic for $y_k \in D(0, \rho + \varepsilon)$ and $u \in D(1, \nu)$. As already mentioned, we have

$$t_1(y_1, u, T) = L(a_0 y_1).$$

This iterative scheme defines a sequence of analytic functions which are consistent with Eq. (4.3). To show this, we shall provide suitable lemmas; prior to those technical details we should establish the connection between $B_T(u)$ and the above. This relation is straightforward, cf. Eq. (3.6):

$$B_T(u) = a_0 u + \sum_{k \geq 1} u^{kT} [y_k^T] \frac{1}{1 - y_k/u} \cdot \frac{B(y_k) - a_0 y_k}{L(B(y_k))} t_k(y_k, u, T).$$

Since $t_k(y_k, u, T)$ is analytic for $y_k \in D(0, \rho + \varepsilon)$, the desired coefficient $[y_k^T]$ is easily evaluated. Providing the expansion

$$\begin{aligned} \frac{B(y_k) - a_0 y_k}{L(B(y_k))} &= \frac{\tau - a_0 \rho}{L(\tau)} \\ &\quad + \left(\frac{b L'(\tau)(\tau - a_0 \rho)}{L(\tau)^2} - \frac{b}{L(\tau)} \right) (1 - y_k/\rho)^{1/2} \\ &\quad + O(1 - y_k/\rho) \end{aligned}$$

for $y_k \to \rho$, we easily obtain

$$\begin{aligned} B_T(u) &= a_0 u + \frac{B(u) - a_0 u}{L(B(u))} u^{-T} \sum_{k \geq 1} t_k(u, u, T) u^{kT} \\ &\quad - \frac{b}{2\sqrt{\pi}} \cdot \frac{(\tau - a_0 \rho) L'(\tau) - L(\tau)}{(1 - \rho/u) L(\tau)^2} T^{-3/2} \rho^{-T} (1 + O(T^{-1/2})) \\ &\quad \cdot \sum_{k \geq 1} t_k(\rho, u, T) u^{kT}. \end{aligned} \tag{4.8}$$

Obviously, the first sum results from a subtracted singularity term for $y_k = u$.

This is why we are interested in infinite sums involving t_k; in order to justify our manipulations, we provide the lemmas promised. Instead of investigating t_k directly, we look at $h_{k-1}(y,u,T) = t_k(y,u,T)u^{(k-1)T}$ in a somewhat generalized manner:

LEMMA 4.3 (Solution of a Recurrence Relation). *Consider*

$$h_n(y,u,T) = f(y,u)h_{n-1}(u,u,T) + g(y,u,T)\psi^T h_{n-1}(\rho,u,T) + r(y,u,T)\psi^T h_{n-1}(u,u,T)$$

for $n \geq 1$, $\rho > 1$ *and* $T \to \infty$. *In addition,*

$$h_0(y,u,T) = h(y).$$

$f(y,u) \not\equiv 0$, $g(y,u,T)$, $r(y,u,T)$ *and* $h(y)$ *denote (multivariate) analytic functions for* $y \in D(0,\rho+\varepsilon)$, $1+\varepsilon > \rho$ *and* $u \in D(1,\nu)$, $1+\nu < \rho$. ψ *is an abbreviation for* $u/\rho < 1$. *If* $\max_{u \in D(1,\nu)}|f(u,u)| = r < 1$, $g(y,u,T) = O(T^m)$, *and* $r(y,u,T) = O(T^l)$ *uniformly for some arbitrary but fixed real* m, l, *we have the uniform bound*

$$|h_n(y,u,T)| \leq CR^n$$

for all $n \geq 0$, $y \in D(0,\rho+\varepsilon)$, $r < R < 1$, *and T sufficiently large.*

Proof. (By Simultaneous Induction). Let $r < R < 1$ arbitrary,

$$C_1 = \max_{u \in D(1,\nu)} |h(u)|,$$

and $C = \max_{u \in D(1,\nu),\, y \in D(0,\rho+\varepsilon)}\{|h(y)|, C_1|f(y,u)|/r|\}$. The implied constants of the $O(\cdot)$-terms concerning $g(y,u,T)$, and $r(y,u,T)$ are denoted by M and L, respectively. We shall show that

$$|h_n(u,u,T)| \leq C_1 R^n,$$
$$|h_n(y,u,T)| \leq CR^n.$$

The case $n = 0$ is trivial; investigating the case $n > 0$ yields

$$\begin{aligned}|h_n(u,u,T)| &\leq rC_1R^{n-1} + MT^m|\psi|^T CR^{n-1} + LT^l|\psi|^T C_1R^{n-1}\\ &= C_1 rR^{N-1} + C_1\left(\frac{CMT^m}{C_1} + LT^l\right)|\psi|^T R^{n-1}\\ &\leq C_1R^n\end{aligned}$$

since $(CMT^m/C_1 + LT^l)|\psi|^T$ can be made less than $R - r > 0$ provided that T is sufficiently large. Similarly,

$$\begin{aligned}|h_n(y,u,T)| &\le |f(y,u)|C_1R^{n-1} + MT^m|\psi|^T CR^{n-1} + LT^l|\psi|^T C_1 R^{n-1}\\ &\le CrR^{n-1} + C\left(MT^m + \frac{C_1LT^l}{C}\right)|\psi|^T R^{n-1}\\ &\le CR^n,\end{aligned}$$

since $(MT^m + C_1LT^l/C)|\psi|^T \le R - r$ for T sufficiently large. ∎

The following lemma justifies Eq. (4.3):

LEMMA 4.4 (Expansion of the Solution of a Recurrence Relation). *With the notations and conditions of Lemma* 4.3 *and the further suppositions* $f(u,u) \ne 0$, $f(\rho,u) \ne 0$, *and* $h(u) \ne 0$ *for* $u \in D(1,\nu)$, *we obtain*

$$h_n(y,u,T) = h_n(\rho,u,T)(1 + O(1 - y/\rho)) \qquad \text{for } y \to \rho$$

uniformly for $u \in D(1,\nu)$, $n \ge 0$, *and* $T \to \infty$.

Proof. By virtue of the Taylor expansion at $y = \rho$ we may write $f(y,u) = f(\rho,u) + f^+(y,u)(1 - y/\rho)$ (and similar for g and r) and obtain

$$\begin{aligned}&\frac{h_n(y,u,T)}{h_n(\rho,u,T)}\\ &= 1 + \frac{f^+(y,u) + g^+(y,u,T)\psi^T h_{n-1}(\rho,u,T)/h_{n-1}(u,u,T) + r^+(y,u,T)\psi^T}{f(\rho,u) + g(\rho,u,T)\psi^T h_{n-1}(\rho,u,T)/h_{n-1}(u,u,T) + r(\rho,u,T)\psi^T}(1 - y/\rho)\\ &= 1 + \frac{f^+(y,u) + O(\psi^T)}{f(\rho,u) + O(\psi^T)}(1 - y/\rho) = 1 + O(1 - y/\rho)\\ &\qquad\qquad \text{for } y \to \rho,\end{aligned}$$

uniformly for $u \in D(1,\nu)$ and $T \to \infty$. To justify the last step we show (by induction) that for all $n \ge 0$

$$\frac{h_n(\rho,u,T)}{h_n(u,u,T)} = O(1)$$

uniformly for $u \in D(1,\nu)$ and $T \to \infty$. The case $n = 0$ is trivial; for $n > 0$

we find

$$\frac{h_n(\rho,u,T)}{h_n(u,u,T)} = \frac{f(\rho,u) + g(\rho,u,T)\psi^T h_{n-1}(\rho,u,T)/h_{n-1}(u,u,T) + r(\rho,u,T)\psi^T}{f(u,u) + g(u,u,T)\psi^T h_{n-1}(\rho,u,T)/h_{n-1}(u,u,T) + r(u,u,T)\psi^T}$$

$$= \frac{f(\rho,u) + O(\psi^T)}{f(u,u) + O(\psi^T)} = O(1).$$

This completes the proof of Lemma 4.4. ■

For $y = u$ and $y = \rho$ let

$$H_y(u,T) = \sum_{n \geq 0} h_n(y,u,T).$$

Both infinite sums of analytic functions obey a uniform bound $K\Sigma_{n \geq 0} R^n = K/(1-R)$ according to Lemma 4.3. Hence, each one represents the limit of a uniformly convergent sequence of analytic functions for $u \in D(1,\nu)$, i.e., is an analytic function itself by virtue of the Theorem of Weierstrass. They are easily evaluated by summing up the recurrence relation for $n \geq 1$,

$$\begin{aligned} H_y(u,T) - h(y) &= f(y,u)H_u(u,T) + g(y,u,T)\psi^T H_\rho(u,T) \\ &\quad + r(y,u,T)\psi^T H_u(u,T) \\ &= v(y,u,T)H_u(u,T) + w(y,u,T)H_\rho(u,T), \end{aligned}$$

where

$$v(y) = v(y,u,T) = f(y,u) + r(y,u,T)\psi^T,$$
$$w(y) = w(y,u,T) = g(y,u,T)\psi^T.$$

We obtain

$$H_u = H_u(u,T) = \frac{w(u)H_\rho(u,T) + h(u)}{1 - v(u)},$$

$$H_\rho = H_\rho(u,T) = \frac{v(\rho)H_u(u,T) + h(\rho)}{1 - w(\rho)}$$

which yields

$$(1 - v(u))H_u = \frac{w(u)v(\rho)}{1 - w(\rho)}H_u + \frac{w(u)h(\rho)}{1 - w(\rho)} + h(u)$$

and finally

$$\begin{aligned} H_u &= \frac{w(u)h(\rho) + h(u)(1 - w(\rho))}{(1 - v(u))(1 - w(\rho)) - w(u)v(\rho)} \\ &= \frac{h(\rho)w(u) + h(u) - h(u)w(\rho)}{1 - v(u) - w(\rho) + v(u)w(\rho) - v(\rho)w(u)} \\ &= \frac{h(\rho)g(u)\psi^T + h(u) - h(u)g(\rho)\psi^T}{\substack{1 - f(u) - r(u)\psi^T - g(\rho)\psi^T + f(u)g(\rho)\psi^T \\ -f(\rho)g(u)\psi^T + O(\psi^{2T})}} \\ &= \frac{h(u)}{1 - f(u)} + \frac{h(u)}{1 - f(u)}\left(\frac{h(\rho)g(u)}{h(u)} + \frac{r(u) + f(\rho)g(u)}{1 - f(u)}\right)\psi^T \\ &\quad + O(\psi^{2T}) \end{aligned}$$

as $T \to \infty$, uniformly for $u \in D(1, \nu)$. To keep the notation simple, we have used the abbreviations $f(y) = f(y, u)$, $g(y) = g(y, u, T)$, $r(y) = r(y, u, T)$, and $\psi = u/\rho$ as usual. Similarly, we find

$$(1 - w(\rho))H_\rho = \frac{v(\rho)w(u)}{1 - v(u)}H_\rho + \frac{v(\rho)h(u)}{1 - v(u)} + h(\rho)$$

and also

$$\begin{aligned} H_\rho &= \frac{h(u)v(\rho) + h(\rho) - h(\rho)v(u)}{1 - v(u) - w(\rho) + v(u)w(\rho) - v(\rho)w(u)} \\ &= h(\rho) + \frac{h(u)f(\rho)}{1 - f(u)} + O(\psi^T) \qquad \text{as } T \to \infty \end{aligned}$$

uniformly for $u \in D(1, \nu)$, also. Note that we need a weaker asymptotic expansion for H_ρ only, cf. Eq. (4.8). Substituting

$$h_k(y_k, u, T) = t_{k+1}(y_{k+1}, u, T)u^{kT}$$

for $k \le 0$ and

$$f(y_k, u) = \alpha(y_k, u),$$

$$g(y_k, u, T) = -\frac{\gamma(y_k, u)}{2\sqrt{\pi}} T^{-3/2} + O(T^{-2}),$$

$$r(y_k, u, T) = O(T^{-2}),$$

$$h(y_1) = L(a_0 y_1)$$

into the recurrence relation of Lemma 4.3 yields (4.7) times $u^{(k-1)T}$, of course, and establishes the connection between (4.8) and H_u, H_ρ:

$$\sum_{k \ge 1} t_k(u, u, T) u^{kT} = u^T H_u,$$

$$\sum_{k \ge 1} t_k(\rho, u, T) u^{kT} = u^T H_\rho. \qquad (4.9)$$

It is worth mentioning that, strictly speaking, both remainder terms $O(T^{-2})$ in the substitution above denote different functions for different indices k; cf. our remark following Eq. (4.7). Thus, we rather should have defined functions $g_k(y_k, u, T)$ and $r_k(y_k, u, T)$ in our previous treatment. However, all (algebraic) operations required are justified for our simplifying assumption due to the uniform estimations, too.

We obtain

$$H_u = \frac{L(a_0 u)}{1 - \alpha(u, u)} - \frac{L(a_0 u)\gamma(u, u)}{2\sqrt{\pi}\,(1 - \alpha(u, u))} \left(\frac{L(a_0 \rho)}{L(a_0 u)} + \frac{\alpha(\rho, u)}{1 - \alpha(u, u)} \right)$$
$$\cdot\, u^T T^{-3/2} \rho^{-T} + O(u^T T^{-2} \rho^{-T})$$

and

$$H_\rho = L(a_0 \rho) + \frac{L(a_0 u)\alpha(\rho, u)}{1 - \alpha(u, u)} + O(u^T T^{-3/2} \rho^{-T}).$$

Mentioning (4.4) and (4.5), it is easy to obtain

$$\frac{1}{1 - \alpha(u, u)} = \left(1 - \frac{L(B(u)) - L(a_0 u)}{L(B(u))} \right)^{-1} = \frac{L(B(u))}{L(a_0 u)}$$

and

$$\frac{L(a_0\rho)}{L(a_0u)} + \frac{\alpha(\rho,u)}{1-\alpha(u,u)} = \frac{L(a_0\rho)}{L(a_0u)} + \frac{L((\rho/u)B(u)) - L(a_0\rho)}{L(B(u))} \cdot \frac{L(B(u))}{L(a_0u)} = \frac{L((\rho/u)B(u))}{L(a_0u)}.$$

In addition, we have

$$\gamma(u,u) = \frac{b(\tau - a_0\rho)}{(1-\rho/u)(\tau - a_0u)L(\tau)} \cdot \left(\frac{L(\tau) - L(a_0u)}{\tau - a_0u} - L'(\tau) - \frac{L(\tau) - L(a_0u)}{\tau - a_0\rho} + \frac{L'(\tau)}{L(\tau)}(L(\tau) - L(a_0u)) \right)$$

$$= \frac{b}{(1-\rho/u)(\tau - a_0u)L(\tau)} \cdot \left(\frac{(L(\tau) - L(a_0u))a_0(u-\rho)}{\tau - a_0u} - \frac{(\tau - a_0\rho)L'(\tau)L(a_0u)}{L(\tau)} \right)$$

and ultimately

$$H_u = L(B(u)) - \frac{L(B(u))\gamma(u,u)L((\rho/u)B(u))}{2\sqrt{\pi}\,L(a_0u)} u^T T^{-3/2}\rho^{-T} + O(u^T T^{-2}\rho^{-T})$$

and

$$H_\rho = L\left(\frac{\rho}{u}B(u)\right) + O(u^T T^{-3/2}\rho^{-T}).$$

Substituting the above in Eq. (4.8) while mentioning (4.9) yields the desired asymptotic expression for $B_T(u)$:

$$\begin{aligned}
B_T(u) &= a_0 u + B(u) - a_0 u \\
&\quad - (B(u) - a_0 u)\frac{\gamma(u,u)L((\rho/u)B(u))}{2\sqrt{\pi}L(a_0 u)} u^T T^{-3/2}\rho^{-T} \\
&\quad - \frac{b((\tau - a_0\rho)L'(\tau) - L(\tau))L((\rho/u)B(u))}{2\sqrt{\pi}(1-\rho/u)L(\tau)^2} u^T T^{-3/2}\rho^{-T} \\
&\quad + O(u^T T^{-2}\rho^{-T}) \\
&= B(u) \\
&\quad - \frac{bL((\rho/u)B(u))}{2\sqrt{\pi}(1-\rho/u)(\tau - a_0 u)L(\tau)} \\
&\quad \cdot \left(\frac{(B(u) - a_0 u)(L(\tau) - L(a_0 u))a_0 u(1-\rho/u)}{L(a_0 u)(\tau - a_0 u)} \right. \\
&\quad \left. + \frac{(\tau - B(u))(\tau - a_0\rho)L'(\tau)}{L(\tau)} - \tau + a_0 u \right) u^T T^{-3/2}\rho^{-T} \\
&\quad + O(u^T T^{-2}\rho^{-T}).
\end{aligned}$$

THEOREM 4.5 (Asymptotic Expansion of $B_T^{(m)}(1)$). *With the notations above, the first few factorial moments of $B_T(u)$ have the asymptotic expansions for $T \to \infty$*

$$\begin{aligned}
B_T(1) &= 1 - \frac{bL(\rho)}{2\sqrt{\pi}(\rho - 1)(\tau - a_0)L(\tau)} \\
&\quad \cdot \left(\frac{(1-a_0)(L(\tau) - L(a_0))a_0(\rho - 1)}{L(a_0)(\tau - a_0)} \right. \\
&\quad \left. + \tau - a_0 - \frac{(\tau - 1)(\tau - a_0\rho)L'(\tau)}{L(\tau)} \right) \\
&\quad \cdot T^{-3/2}\rho^{-T} + O(T^{-2}\rho^{-T}), \\
B_T'(1) &= \frac{1}{1 - P'(1)} + O(T^{-1/2}\rho^{-T}), \\
B_T^{(m)}(1) &= O(1) \qquad \textit{for } m \textit{ arbitrary but fixed},
\end{aligned}$$

where $B(z)$ denotes the solution of $B(z) = zP(B(z))$ and

$$P(z) = A(L(z))$$

$$b = \sqrt{\frac{2P(\tau)}{P''(\tau)}}.$$

Proof. The expression for $B_T(1)$ is straightforward. Since, roughly speaking, the derivation of an asymptotic expression is permitted if its domain of validity lies within the complex plane, the necessary derivatives of $B_T(u)$ are most easily obtained: As frequently mentioned, we have a remainder term which represents a function analytic for $u \in D(1, \nu)$.

The result for $B_T'(1) = B'(1) + O(T^{-1/2}\rho^{-T})$ follows by differentiating $B(z) = zP(B(z))$ w.r.t. z. ■

Note that the remainder in the asymptotic expressions of $B_T(1)$ and $B_T'(1)$ might be improved to $O(T^{-5/2}\rho^{-T})$ and $O(T^{-3/2}\rho^{-T})$, respectively, due to our remark on equation (4.1)!

5. Final Results

Now we are able to return to the PGF of SRD(T), which has been evaluated as

$$S_T(u) = \sum_{k \geq 0} s_{k,T} u^k = \frac{1 - B_T(1)}{1 - B_T(u)},$$

cf. Eq. (3.2). We shall investigate the moments of this distribution, i.e., the quantities

$$E^n(T) = E\left[\mathrm{SRD}(T)^n\right] = \sum_{k \geq 0} k^n s_{k,T}.$$

In addition, we define the nth factorial moment by

$$F^n(T) = \sum_{k \geq 0} [k]_n s_{k,T} = S_T^{(n)}(1),$$

where $[k]_n = k(1-1) \cdots (k-n+1)$ denotes the falling factorial. Note that n is assumed to be fixed; all $O(\;)$-terms are uniform in T only. Since

$[k]_n = k^n + O(k^{n-1})$, we obtain

$$F^n(T) = E^n(T) + O(E^{n-1}(T)).$$

If we could provide $F^{n-1}(T) = O(F^n(T))$, a simple induction argument shows that

$$E^n(T) = F^n(T) + O(F^{n-1}(T)); \tag{5.1}$$

hence it seems reasonable to investigate the factorial moments. We have

$$S_T(z) = g(B_T(z))$$

for $g(z) = (1 - B_T(1))/(1 - z)$. An easy computation shows that

$$g^{(j)}(z)\big|_{z=B_T(1)} = g^{(j)}(B_T(1)) = \frac{j!}{(1 - B_T(1))^j}$$

for all $j \geq 0$. Using the formula of Faà di Bruno (cf. [5, p. 50]),

$$\begin{aligned}(b(a(z)))^{(n)}\big|_{z=t} &= \sum_{j=0}^{n} b^{(j)}(a(t)) \sum_{\substack{k_1+k_2+\cdots+k_n=j\\ k_1+2k_2+\cdots+nk_n=n\\ k_i\geq 0}} \frac{n!}{k_1!(1!)^{k_1}\cdots k_n!(n!)^{k_n}}\\ &\quad\cdot (a^{(1)}(t))^{k_1}\cdots(a^{(n)}(t))^{k_n},\end{aligned}$$

we are able to express $S_T^{(n)}(1)$ in terms of $g^{(j)}(B_T(1))$ and $B_T^{(j)}(1)$; setting $b(z) = g(z)$, $a(z) = B_T(z)$, and $t = 1$, we find that

$$\begin{aligned}S_T^{(n)}(1) = \sum_{j=0}^{n} \frac{1}{(1 - B_T(1))^j} \sum_{\substack{k_1+\cdots+k_n=j\\ k_1+\cdots+nk_n=n\\ k_i\geq 0}} c_{j,n,k_1,k_2,\ldots,k_n}\\ \cdot (B_T^{(1)}(1))^{k_1}\cdots(B_T^{(n)}(1))^{k_n}\end{aligned}$$

with the abbreviation

$$c_{j,n,k_1,k_2,\ldots,k_n} = j!\frac{n!}{k_1!(1!)^{k_1}\cdots k_n!(n!)^{k_n}}.$$

Using the fact that $B_T^{(m)}(1) = O(1)$ for $m \geq 0$ from Theorem 4.5, an overall contribution of the inner sum of $O(1)$ may be found. Because

$$1 - B_T(1) = O(T^{-3/2}\rho^{-T}),$$

the major contributions come from $(1 - B_T(1))^{-j}$ with $j = n$. Hence, we may discard all terms of the outer sum concerning $S_T^{(n)}(1)$ except for $j = n$; i.e., we obtain

$$\begin{aligned} S_T^{(n)}(1) &= \sum_{\substack{k_1+\cdots+k_n=n \\ k_1+\cdots+nk_n=n \\ k_i \geq 0}} \frac{c_{n,n,k_1,k_2,\ldots,k_n}}{(1 - B_T(1))^n}(B'_T(1))^{k_1}\cdots(B_T^{(n)}(1))^{k_n} \\ &\quad + O(T^{3(n-1)/2}\rho^{(n-1)T}) \\ &= n!\left(\frac{B'_T(1)}{1 - B_T(1)}\right)^n + O(T^{3(n-1)/2}\rho^{(n-1)T}), \end{aligned} \tag{5.2}$$

since the conditions concerning the inner sum hold for $k_1 = n$ only. Substituting the expansion above, we find

$$F^n(T) = S_T^{(n)}(1) = O(T^{3n/2}\rho^{nT}).$$

Since $F^{n-1}(T) = O(F^n(T))$, the condition for Eq. (5.1) is justified and we may conclude that

$$E^n(T) = F^n(T) + O(F^{n-1}(T)).$$

The remainder above disappears within the remainder term established for $S_T^{(n)}(1)$; hence our final result follows:

THEOREM 5.1. *With the conditions* (i)–(vi) *from Section* 4 *and the notations above, the nth moment* (*n arbitrary but fixed*) *of* SRD(T) *fulfills*

$$E^n(T) = n!\mu(T)^n(1 + O(T^{-1/2}))$$

with

$$\mu(T) = \frac{2\sqrt{\pi}(\rho-1)(\tau-a_0)L(\tau)}{bL(\rho)(1-P'(1))} \cdot \left(\frac{(1-a_0)(L(\tau)-L(a_0))a_0(\rho-1)}{L(a_0)(\tau-a_0)} + \tau - a_0 - \frac{(\tau-1)(\tau-a_0\rho)L'(\tau)}{L(\tau)}\right)^{-1} T^{3/2}\rho^T,$$

where $P(z) = \mathrm{A}(\mathrm{L}(z))$, $\rho = \tau/\mathrm{P}(\tau)$, *and* $b = \sqrt{2P(\tau)/P''(\tau)}$.

Note that the remainder $1 + O(T^{-1/2})$ springs from $(B'_T(1)/(1 - B_T(1)))^n$; it causes the remainder of (5.2) to disappear.

Conclusions

This paper contains a detailed analysis of the successful run duration SRD(T) of a discrete time single server system with nonpreemptive LCFS task scheduling. SRD(T) is closely related to the ability of this system to meet the fixed deadlines T of all tasks arriving at the system, from the time it is turned on to the year 9999, for example. It extends our analysis of preemptive LCFS scheduling (cf. [2]) and FCFS scheduling (cf. [8]) to the case of the nonpreemptive LCFS scheduling discipline. Again, we have found impressive results concerning the expectation of SRD(T), unfortunately weakened by a large standard deviation; see [9] for a more detailed discussion.

Comparing nonpreemptive LCFS to FCFS scheduling shows significantly better deadline meeting behavior of the latter. On the other hand, nonpreemptive LCFS and preemptive LCFS are more difficult to compare; it is devoted to a forthcoming paper. Note, however, that those results are the same for $L(z) = z$, i.e., constant task execution times of 1 cycle.

To establish our results we have used a coefficient extraction technique for multivariate functions which we call *asymptotic separation*: using a slight extension of well-known asymptotic techniques it is possible to separate multivariate analytic functions. We feel that this method is of independent interest and should be useful in the case of investigating sequences of random variables X_k, which are in some sense "weakly dependent." In our case, we had to deal with random variables having a Markov-like property, but asymptotic separation is not restricted to this case. Note however, that queueing theory provides no solution to our problem, because we are forced to study nonequilibrium behavior in order to obtain our desired quantities.

References

1. E. A. Bender, Asymptotic methods in enumeration, *SIAM Rev.* **16** (1974), 485–515.
2. J. Blieberger and U. Schmid, Preemptive LCFS scheduling in hard real time applications, *Performance Evaluation* **15** (1992), 203–215.
3. Ph. Flajolet and J. S. Vitter, "Handbook of Theoretical Computer Science," North-Holland, Amsterdam, 1989.
4. Ph. Flajolet and A. Odlyzko, Singularity analysis of generating functions, *SIAM J. Discrete Math.* **3** (1990), 216–240.
5. D. E. Knuth, "The Art of Computer Programming. Volume 1. Fundamental Algorithms," 2nd ed., Addison–Wesley, Reading, MA, 1973.
6. A. Meir and J. W. Moon, On the altitude of nodes in random trees, *Canad. Math. J.* **30** (1978), 997–1015.
7. A. Meir and J. W. Moon, On an asymptotic method in enumeration, *J. Combin. Theory Ser. A* **51** (1989), 77–89.
8. U. Schmid and J. Blieberger, Some investigations on FCFS scheduling in hard real time applications, *Jornal of Computers and System Sciences* **45** (1992), 493–512.
9. M. Drmota and U. Schmid, Exponential limiting distributions in queueing systems with deadlines, *SIAM J. Appl. Math.* **53**(1) (1993), 301–318.